2D- Geometric Transformation

Two Dimensional Transformation

Translation

Translation is a process of changing the position of an object in a straight-line path from one coordinate location to another. We can translate a two dimensional point by adding translation distances, t_x and t_y, to the original coordinate position (x, y) to move the point to a new position (x′, y′), as shown in the Fig. 4.1.

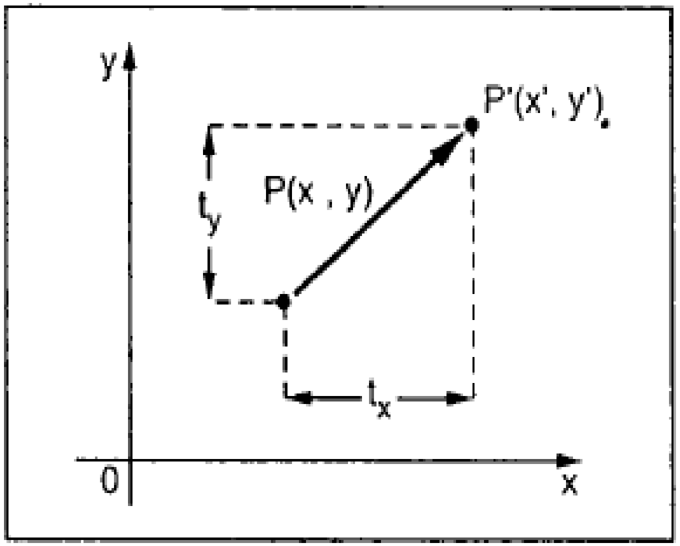


Fig. 4.1

$$x' = x + t_x \quad \ldots (4.1)$$

$$y' = y + t_y \quad \ldots (4.2)$$

The translation distance pair (t_x, t_y) is called a **translation vector** or **shift vector**.

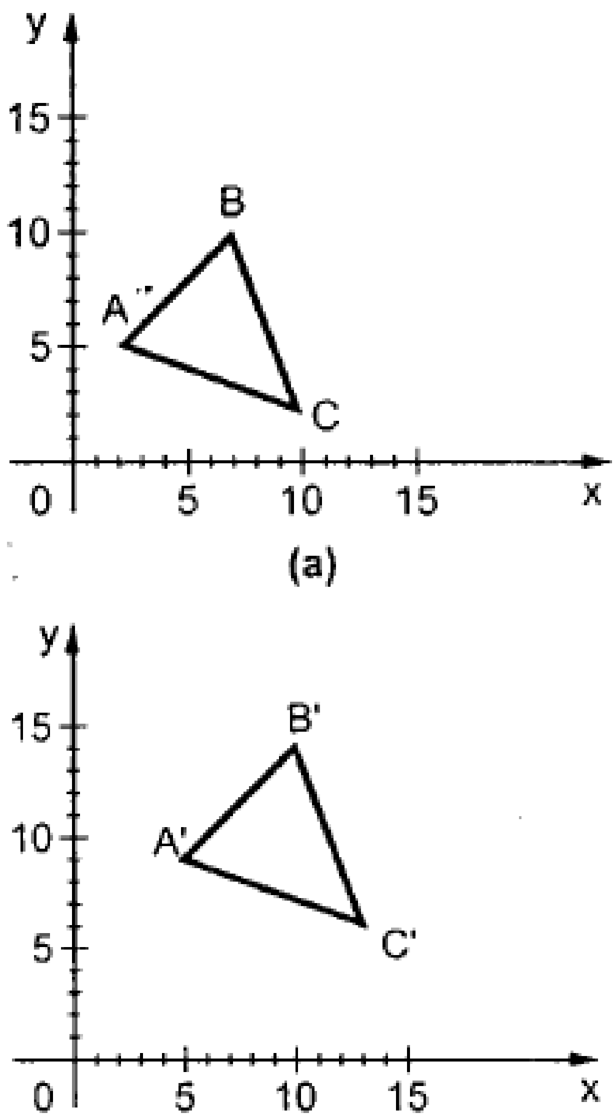
It is possible to express the translation equations 4.1 and 4.2 as a single matrix equation by using column vectors to represent coordinate positions and the translation vector :

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

This allows us to write the two dimensional translation equations in the matrix form :

$$P' = P + T \quad \ldots (4.3)$$

Translate a polygon with coordinates A (2, 5), B (7, 10) and C (10, 2) by 3 units in x direction and 4 units in y direction.



(a)

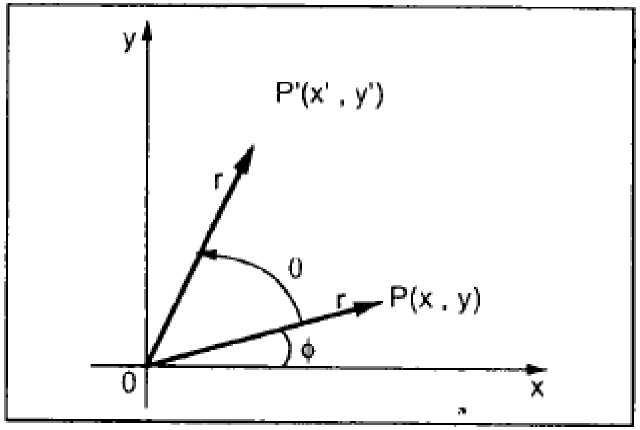
(b)

$$A' = A + T = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$B' = B + T = \begin{bmatrix} 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$C' = C + T = \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

Rotation



Let us consider the rotation of the object about the origin, as shown in the Fig. 4.3.

Here, r is the constant distance of the point from the origin, angle ϕ is the original angular position of the point from the horizontal, and θ is the rotation angle. Using standard trigonometric equations, we can express the transformed coordinates in terms of angles θ and ϕ as

Compound Angle

$$\left.\begin{aligned} x' &= r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' &= r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{aligned}\right\} \quad \ldots(4.4)$$

The original coordinates of the point in polar coordinates are given as

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \quad \ldots(4.5)$$

Substituting equations 4.5 into 4.4, we get the transformation equations for rotating a point (x, y) through an angle θ about the origin :

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}\right\} \quad \ldots(4.6)$$

The above equations can be represented in the matrix form as given below

$$[x' \;\; y'] = [x \;\; y]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad P' = P \cdot R \quad \ldots(4.7)$$

where R is rotation matrix and it is given as

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \ldots(4.8)$$

It is important to note that positive values for the rotation angle define counterclockwise rotations about the rotation point and negative values rotate objects in the clockwise sense.

For negative values of θ i.e., for clockwise rotation, the rotation matrix becomes

$$R = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \because \cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta \quad \ldots(4.9)$$

A point (4, 3) is rotated counterclockwise by an angle of 45°. Find the rotation matrix and the resultant point.

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \quad P' = [4 \;\; 3] \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4/\sqrt{2} - 3/\sqrt{2} & 4/\sqrt{2} + 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}$$

Scaling

A scaling transformation changes the size of an object. This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors S_x and S_y to produce the transformed coordinates (x′, y′).

$$x' = x \cdot S_x$$

$$\text{and} \quad y' = y \cdot S_y \qquad \ldots (4.10)$$

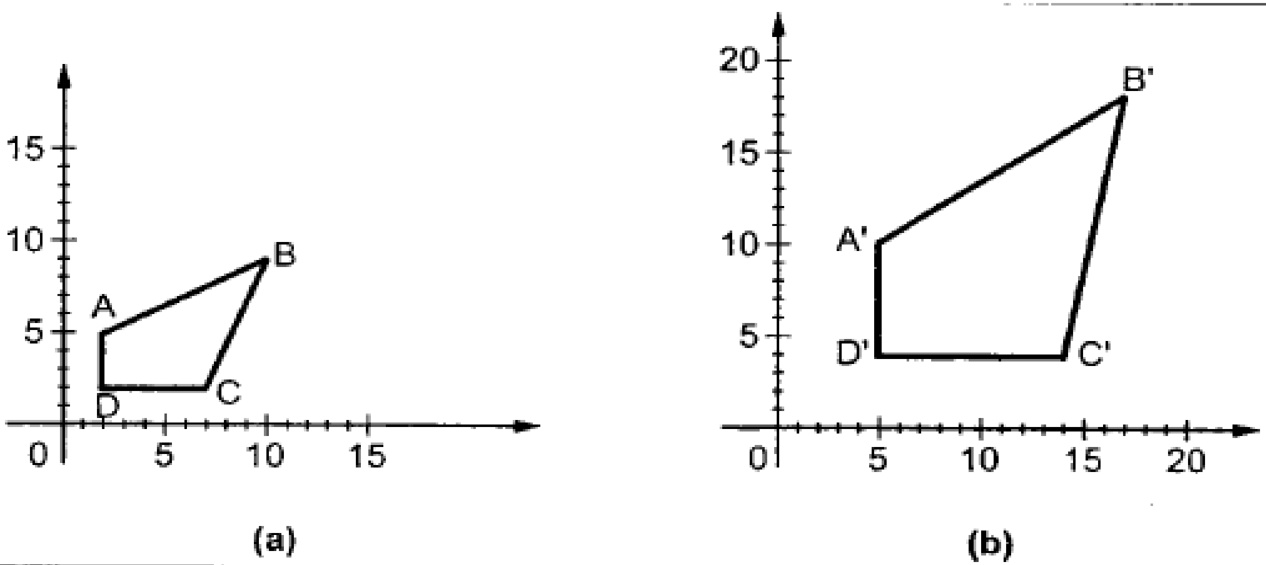


Fig. 4.4

$$[x' \;\; y'] = [x \;\; y] \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

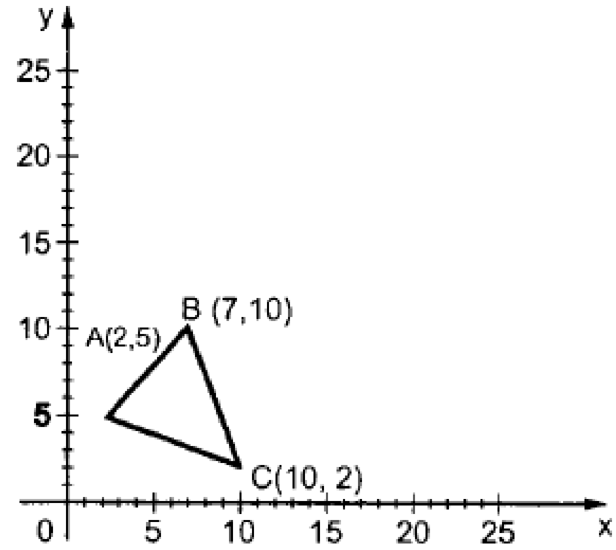
$$= [x \cdot S_x \quad y \cdot Sy] \qquad \ldots (4.11)$$

$$= P \cdot S$$

Scale the polygon with coordinates A (2, 5), B (7, 10) and C (10, 2) by two units in x direction and two units in y direction.

Here $S_x = 2$ and $S_y = 2$. Therefore, transformation matrix is given as

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



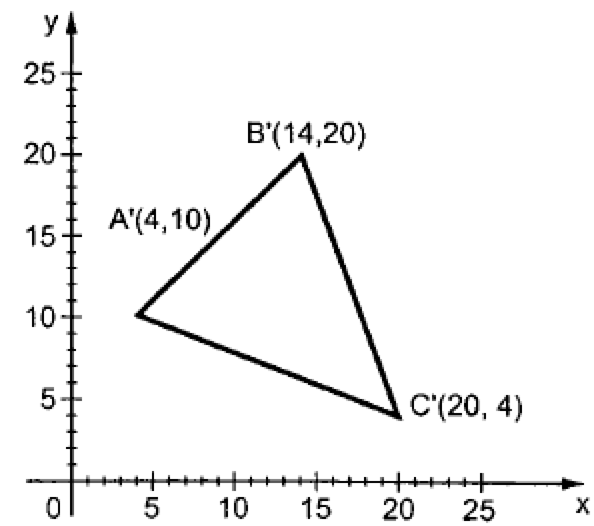
(a) Original object

ject matrix is :

$$\begin{matrix} & x & y \\ A & 2 & 5 \\ B & 7 & 10 \\ C & 10 & 2 \end{matrix}$$

$$\begin{matrix} A' \\ B' \\ C' \end{matrix} \begin{bmatrix} x_1' & y_1' \\ x_2' & y_2' \\ x_3' & y_3' \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 14 & 20 \\ 20 & 4 \end{bmatrix}$$



(b) Scaled object

Home Work

- Translate a triangle ABC by 5 units in x direction where co-ordinates are A(5,5), B(10,5), C(10,10).
- Scale the polygon with co-ordinates A(2,5), B(7,10), C(10,2) by two units in x direction and two units in y direction.
- Scale the triangle ABC to reduce it to half of its size where co-ordinates of triangle are A=(5,5), B=(10,5), C=(10,10)
- Scale same above triangle ABC to extended it to double of its size only in x-direction.

Homogeneous Coordinates

$$P' = P \cdot M_1 + M_2$$

For translation :

$$P' = P \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

i.e. M_1 = Identity matrix

M_2 = Translation vector

For rotation :

$$P' = P \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. M_1 = Rotational matrix

$M_2 = 0$

For scaling :

$$P' = P \cdot \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. M_1 = Scaling matrix

$M_2 = 0$

4.3.1 Homogeneous Coordinates for Translation

The homogeneous coordinates for translation are given as

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

Therefore, we have

$$[x' \ y' \ 1] = [x \ y \ 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$= [x + t_x \quad y + t_y \quad 1]$$

4.3.2 Homogeneous Coordinates for Rotation

The homogeneous coordinates for rotation are given as

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have

$$[x' \quad y' \quad 1] = [x \quad y \quad 1] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [x\cos\theta - y\sin\theta \quad x\sin\theta + y\cos\theta \quad 1]$$

4.3.3 Homogeneous Coordinates for Scaling

The homogeneous coordinate for scaling are given as

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have

$$[x' \quad y' \quad 1] = [x \quad y \quad 1] \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [x \cdot S_x \quad y \cdot S_y \quad 1]$$

Ex. 4.4 : *Give a 3 × 3 homogeneous coordinate transformation matrix for each of the following translations*

a) Shift the image to the right 3-units

b) Shift the image up 2 units

c) Move the image down ½ unit and right 1 unit

d) Move the image down 2/3 unit and left 4 units

Sol. : We know that homogenous coordinates for translation are

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

a) Here, $t_x = 3$ and $t_y = 0$

$$\therefore \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

b) Here, $t_x = 0$ and $t_y = 2$

$$\therefore \qquad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

c) Here, $t_x = 1$ and $t_y = -0.5$

$$\therefore \qquad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}$$

d) Here, $t_x = -4$ and $t_y = -0.66$

$$\therefore \qquad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -0.66 & 1 \end{bmatrix}$$

Ex. 4.5 : *Find the transformation matrix that transforms the given square ABCD to half its size with centre still remaining at the same position. The coordinates of the square are : A(1, 1), B (3, 1), C (3, 3), D (1, 3) and centre at (2, 2). Also find the resultant coordinates of square.*

Sol. : This transformation can be carried out in the following steps.

1. Translate the square so that its center coincides with the origin.
2. Scale the square with respect to the origin.
3. Translate the square back to the original position.

Thus, the overall transformation matrix is formed by multiplication of three matrices.

$$\therefore \qquad T_1 \cdot S \cdot T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$
= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1.5 & 1.5 & 1 \\ 2.5 & 1.5 & 1 \\ 2.5 & 2.5 & 1 \\ 1.5 & 2.5 & 1 \end{bmatrix}
$$

Ex. 4.6 : *Find a transformation of triangle A (1, 0), B (0, 1), C (1, 1) by*

a) Rotating 45° about the origin and then translating one unit in x and y direction.

b) Translating one unit in x and y direction and then rotating 45° about the origin.

Sol. : The rotation matrix is

$$R = \begin{bmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

The translation matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R \cdot T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} + 1 & \frac{1}{\sqrt{2}} + 1 & 1 \\ -\frac{1}{\sqrt{2}} + 1 & \frac{1}{\sqrt{2}} + 1 & 1 \\ 1 & \sqrt{2} + 1 & 1 \end{bmatrix}$$

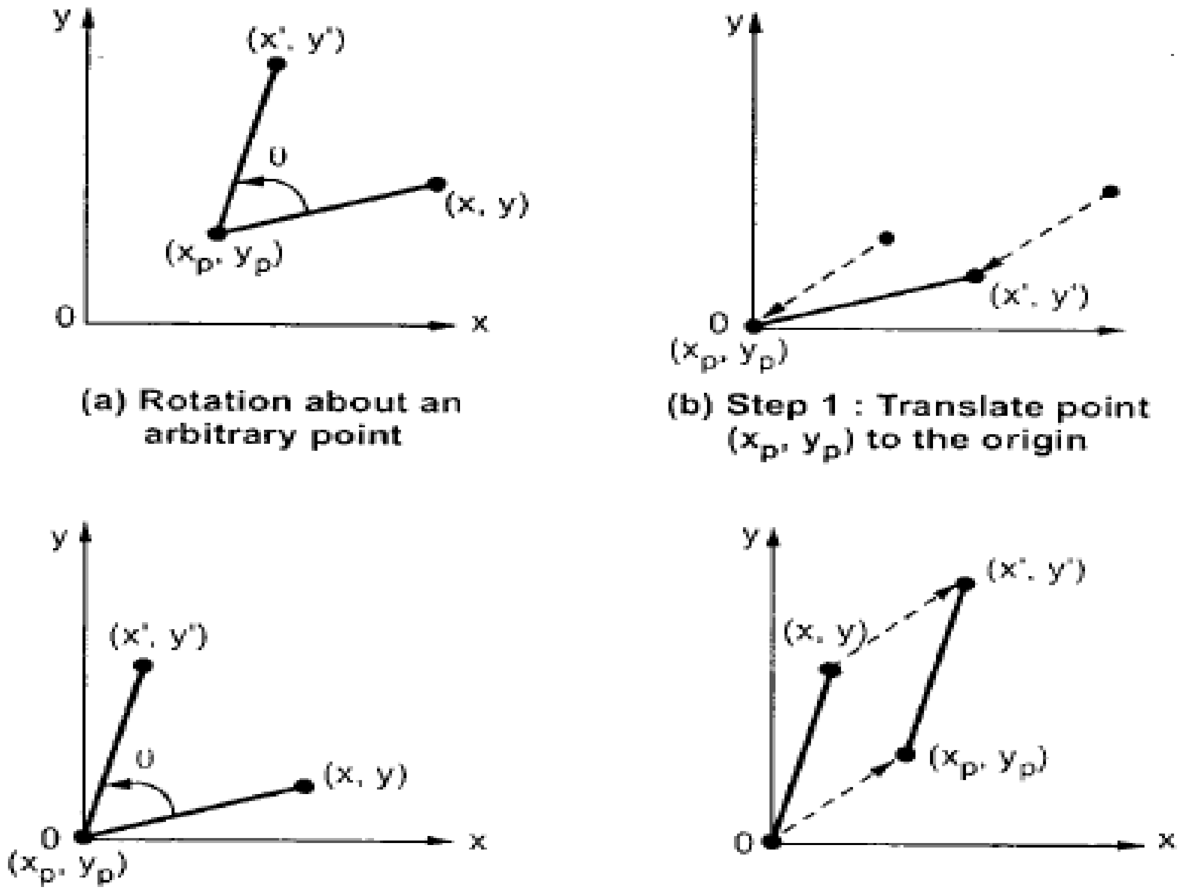
b)

$$T \cdot R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{2} & 1 \\ -1/\sqrt{2} & 3/\sqrt{2} & 1 \\ 0 & 2\sqrt{2} & 1 \end{bmatrix}$$

Rotation About an Arbitrary Point



(a) Rotation about an arbitrary point

(b) Step 1 : Translate point (x_p, y_p) to the origin

(c) Step 2 : Rotate it about the origin

(d) Step 3 : Translate back to the original position

To rotate an object about an arbitrary point, (x_p, y_p) we have to carry out three steps :

1. Translate point (x_p, y_p) to the origin
2. Rotate it about the origin and
3. Finally, translate the center of rotation back where it belongs (See Fig.4.6)

The translation matrix to move point (x_p, y_p) to the origin is given as

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_p & -y_p & 1 \end{bmatrix}$$

The rotation matrix for counterclockwise rotation of point about the origin is given as

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation matrix to move the center point back to its original position is given as

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

Therefore, the overall transformation matrix for a counterclockwise rotation by an angle θ about the point (x_p, y_p) is given as

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_p & -y_p & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_p\cos\theta + y_p\sin\theta & -x_p\sin\theta - y_p\cos\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_p\cos\theta + y_p\sin\theta + x_p & -x_p\sin\theta - y_p\cos\theta + y_p & 1 \end{bmatrix} \quad \ldots (4.18)$$

Ex. 4.7 : *Perform a counterclockwise 45° rotation of triangle A (2, 3), B (5, 5), C (4, 3) about point (1, 1).*

Sol. : From equation 4.18 we have

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_p \cos\theta + y_p \sin\theta + x_p & -x_p \sin\theta - y_p \cos\theta + y_p & 1 \end{bmatrix}$$

Here, $\theta = 45°$, $x_p = 1$ and $y_p = 1$. Substituting values we get

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & -\sqrt{2}+1 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 5 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & -\sqrt{2}+1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}}+1 & \frac{3}{\sqrt{2}}+1 & 1 \\ 1 & \frac{8}{\sqrt{2}}+1 & 1 \\ \frac{1}{\sqrt{2}}+1 & \frac{5}{\sqrt{2}}+1 & 1 \end{bmatrix}$$

Reflection

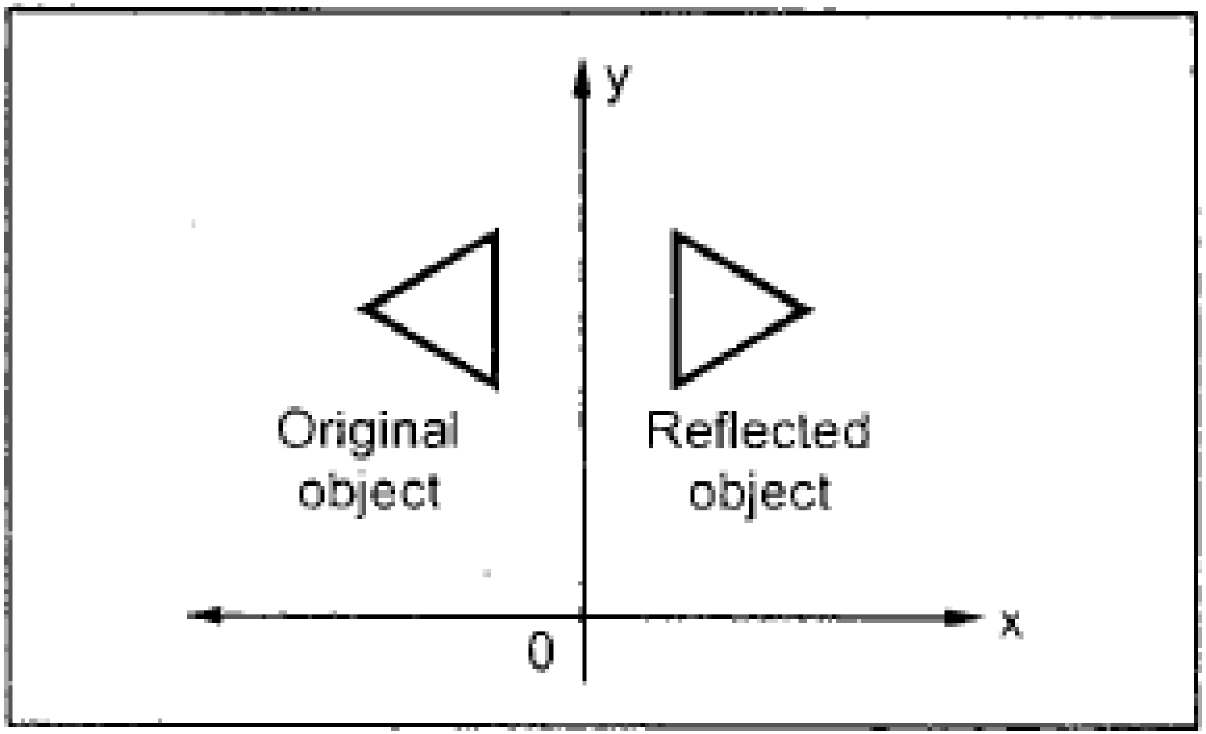
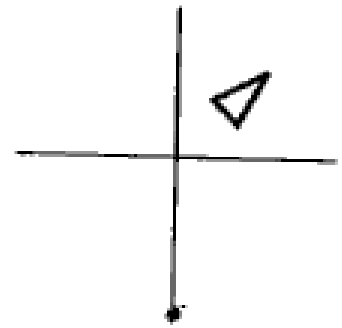
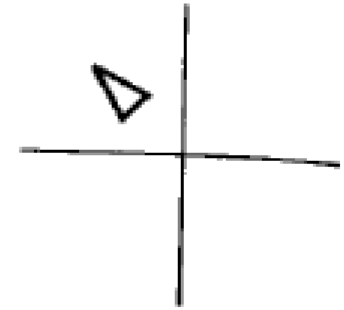
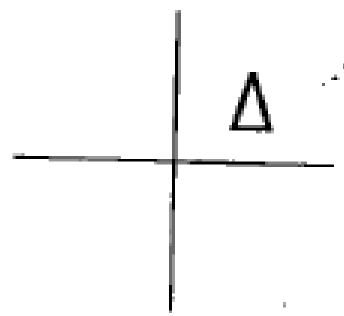
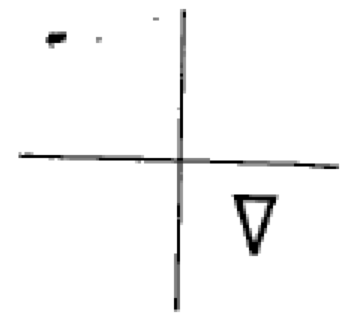
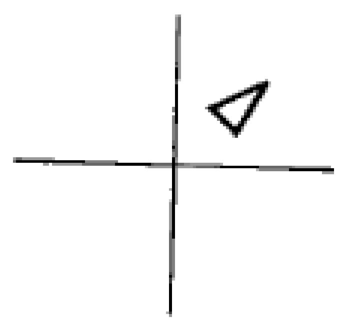
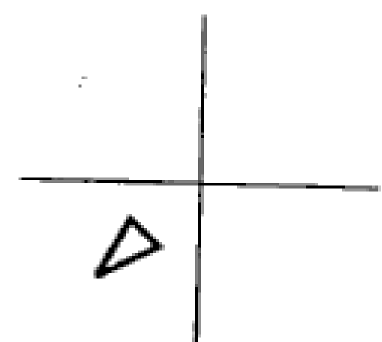
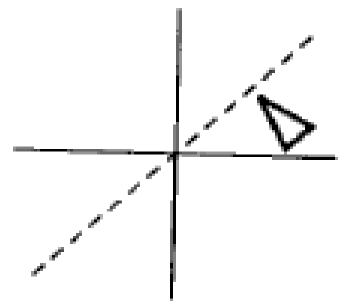
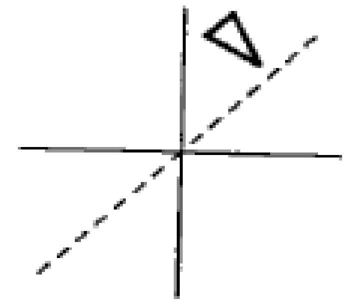
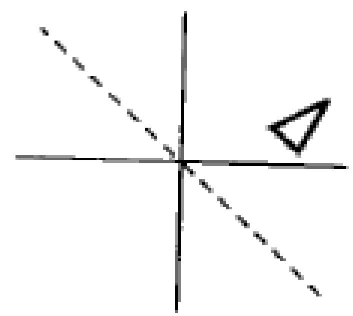
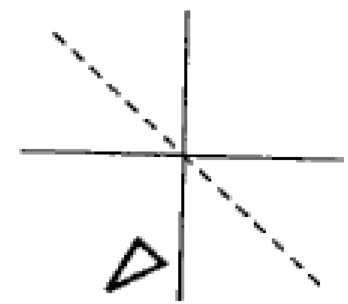
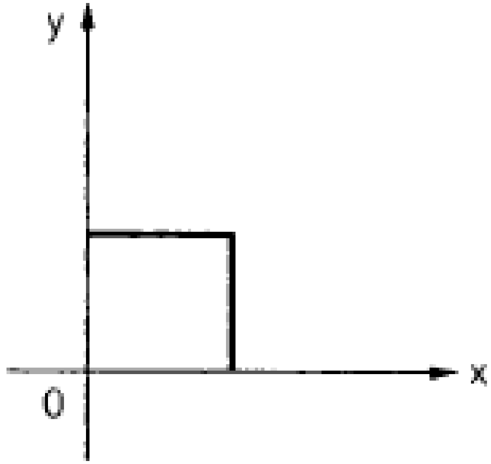


Fig. 4.7 Reflection about y axis

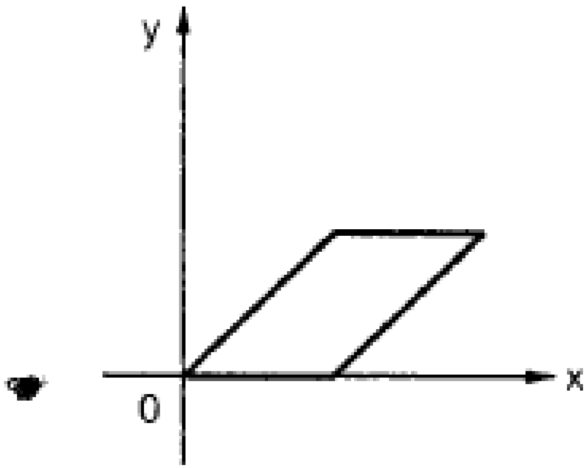
Reflection	Transformation matrix	Original image	Reflected image
Reflection about Y-axis	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about X axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about origin	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

Reflection about line $y = x$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Reflection about line $y = -x$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

Shear



(a) Original object



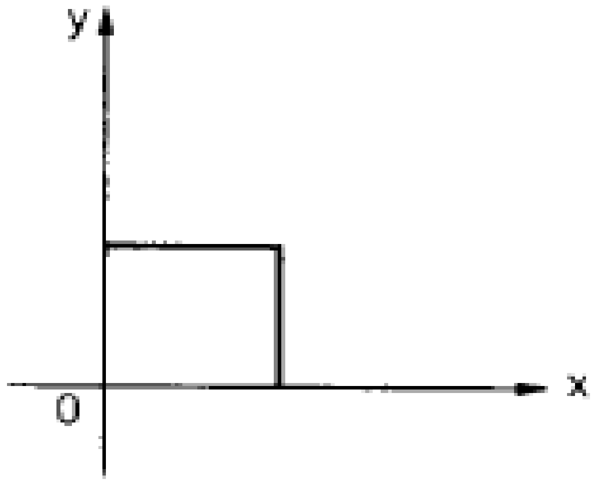
(b) Object after x shear

$$X\_sh = \begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

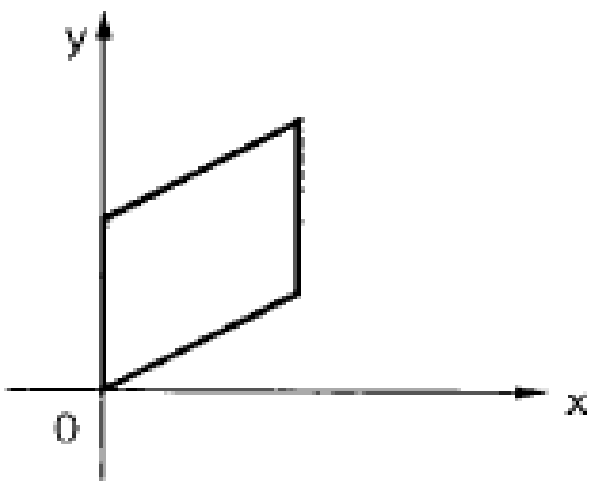
$$x' = x + Sh_x \cdot y \qquad \text{and}$$

$$y' = y$$

Shear (contd…)



(a) Original object



(b) Object after y shear

The transformation matrix for y shear is given as

$$Y\_sh = \begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad x' = x \text{ and } \quad y' = y + Sh \cdot x$$

Shearing Relative to Other Reference Line

$$\text{x shear with y reference line} : \begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ -Sh_x \cdot y_{ref} & 0 & 1 \end{bmatrix}$$

$$\text{y shear with x reference line} : \begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & -Sh_y \, x_{ref} & 0 \end{bmatrix}$$

Apply the shearing transformation to square with A(0, 0), B (1, 0), C (1, 1) and D (0, 1) as given below

a) Shear parameter value of 0.5 relative to the line $y_{ref} = -1$

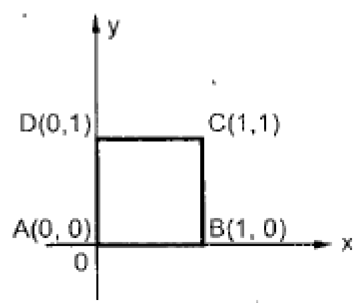
b) Shear parameter value of 0.5 relative to the line $x_{ref} = -1$

a) Here $Sh_x = 0.5$ and $y_{ref} = -1$

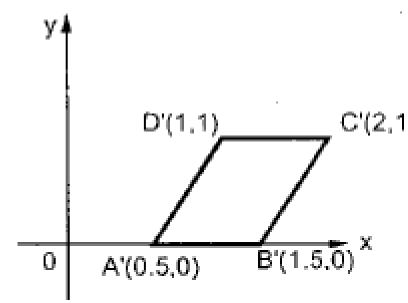
$$\therefore \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ -Sh_x \cdot y_{ref} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 1 \\ 1.5 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



(a) Original square

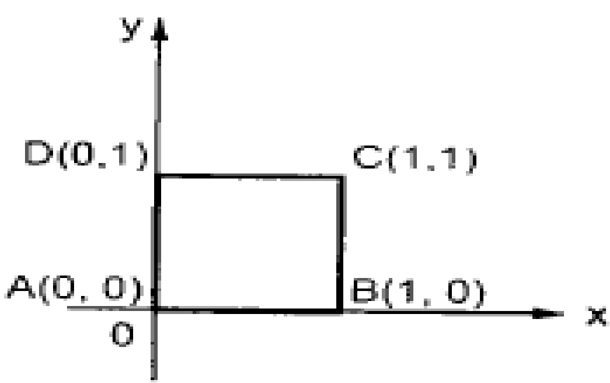


(b) Sheared square

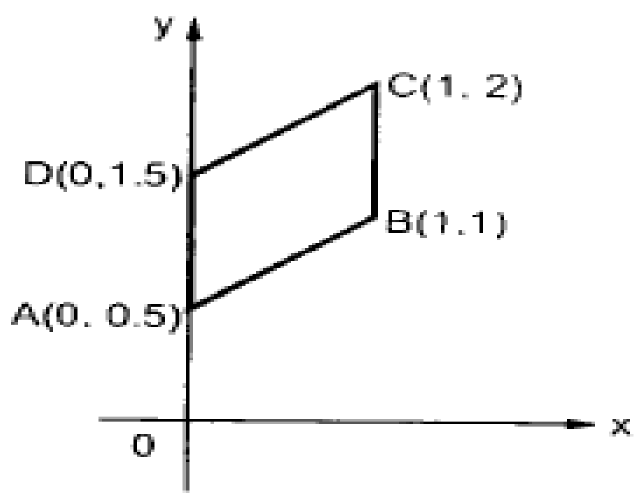
b) Here $Sh_y = 0.5$ and $x_{ref} = -1$

$$\therefore \quad \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & -Sh_y \cdot x_{ref} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1.5 & 1 \end{bmatrix}$$



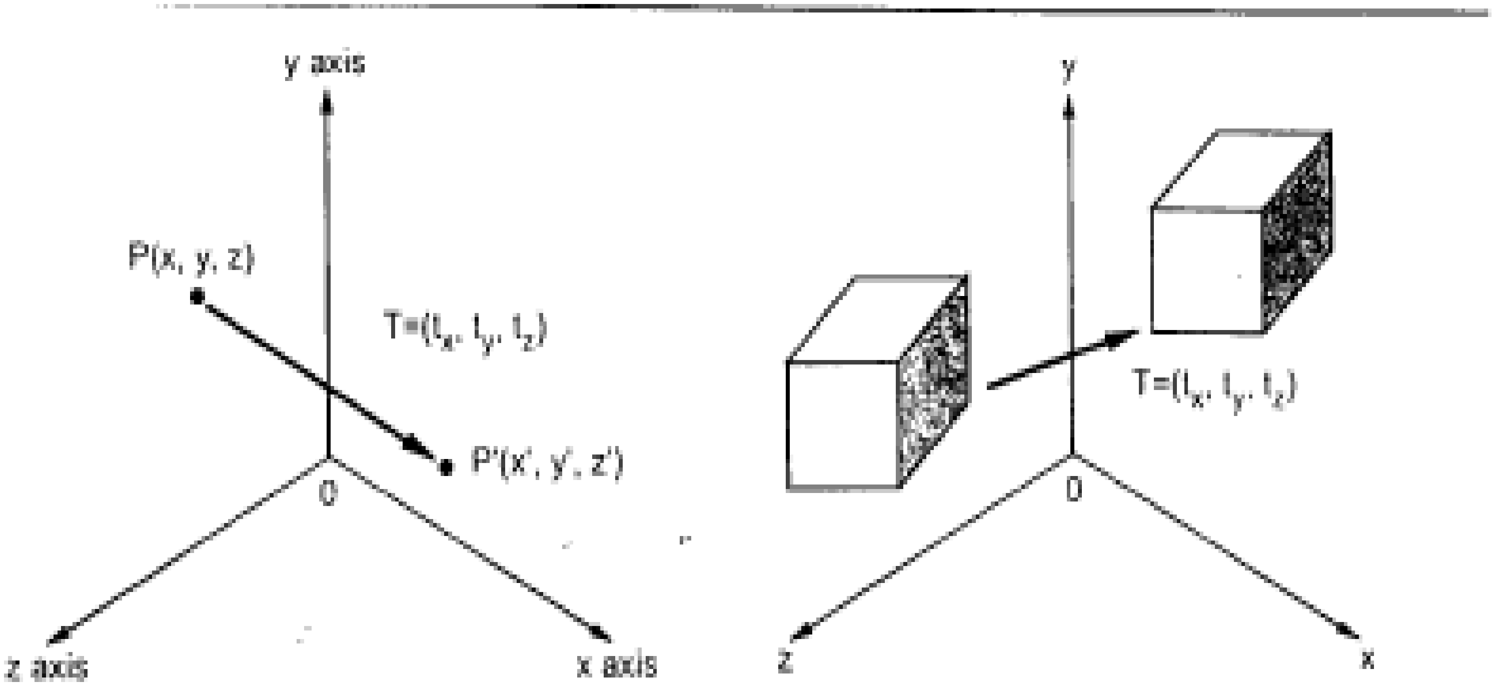
(a) Original square



(b) Sheared square

3D Transformation

Translation



(a) Translating point

(b) Translating object

Translation (contd…)

Three dimensional transformation matrix for translation with homogeneous coordinates is as given below. It specifies three coordinates with their own translation factor.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

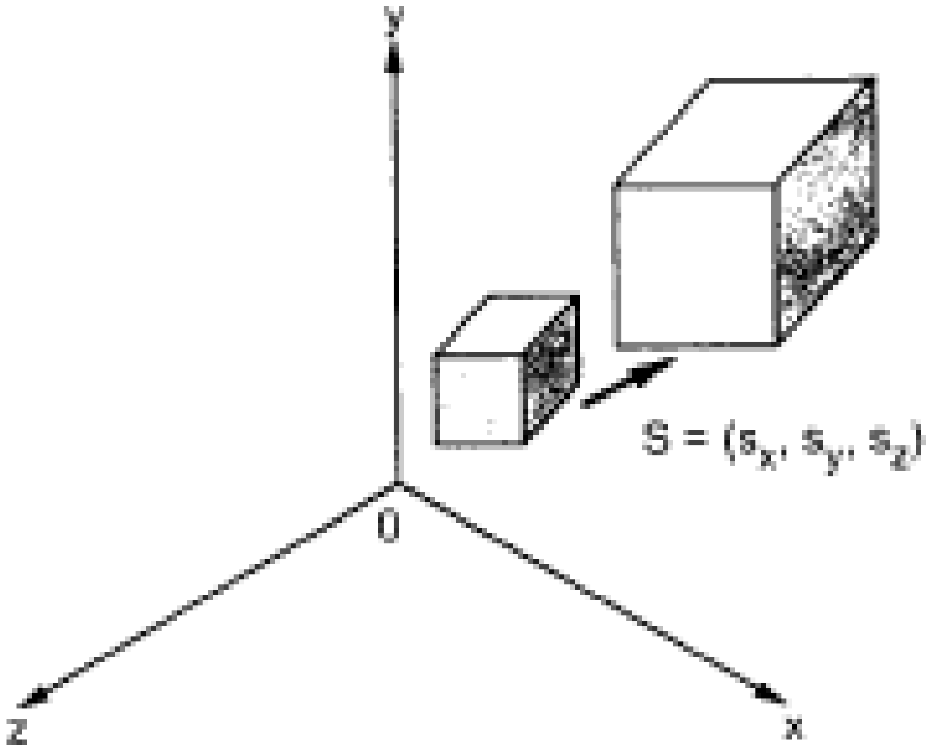
$$\therefore \quad P' = P \cdot T$$

$$\therefore \; [x' \;\; y' \;\; z' \;\; 1] = [x \;\; y \;\; z \;\; 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

$$= [x + t_x \quad y + t_y \quad z + t_z \quad 1] \qquad \ldots (6.1)$$

Like two dimensional transformations, an object is translated in three dimensions by transforming each vertex of the object.

Scaling



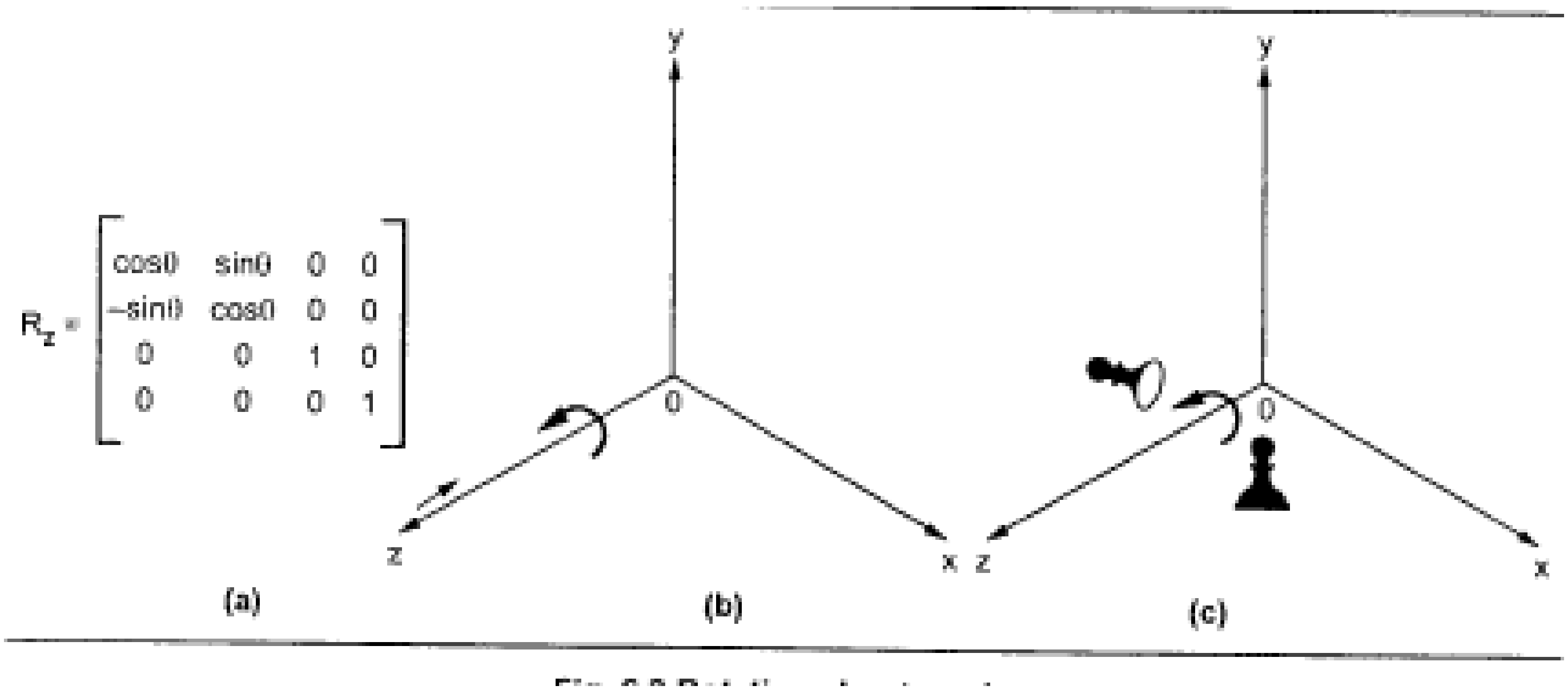
Scaling (contd…)

It specifies three coordinates with their own scaling factor.

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad P' = P \cdot S$$

Rotation



$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)

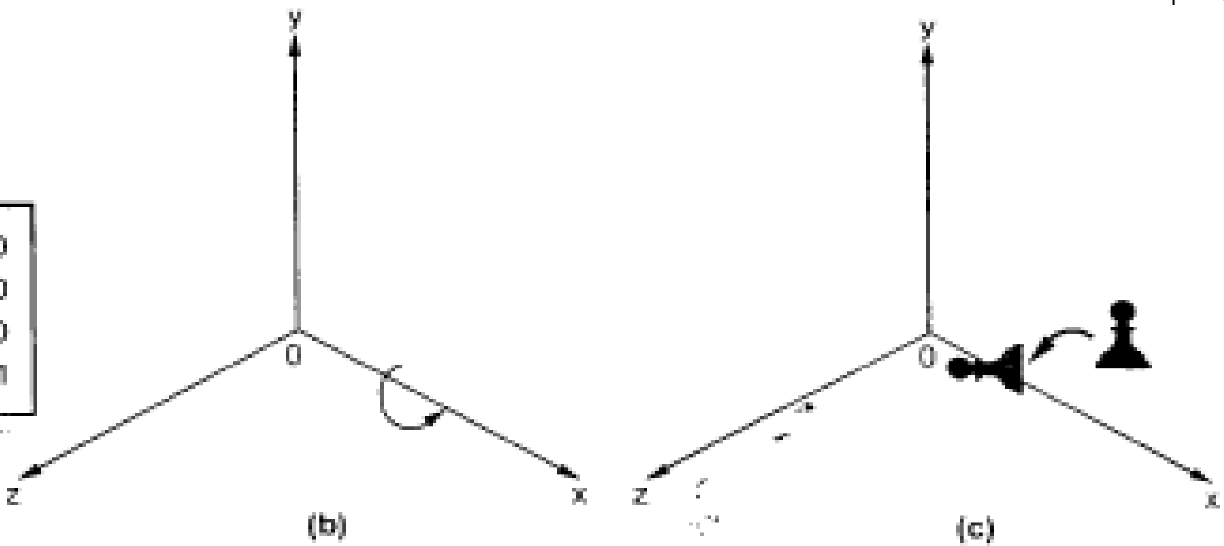


Fig. 6.4 Rotation about x axis

$$R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)

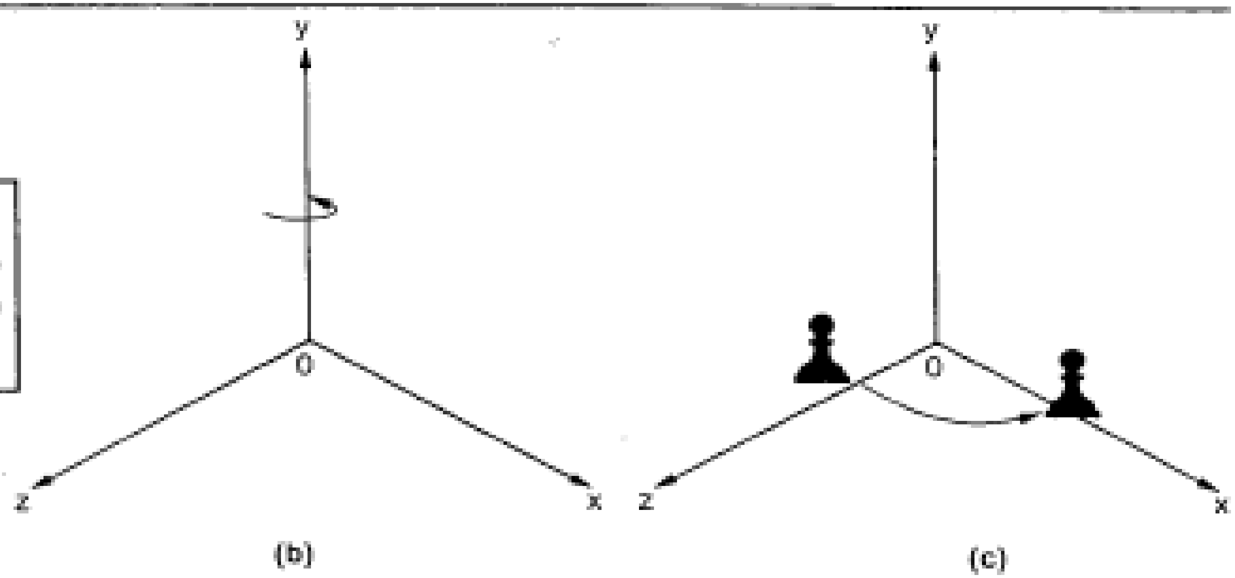
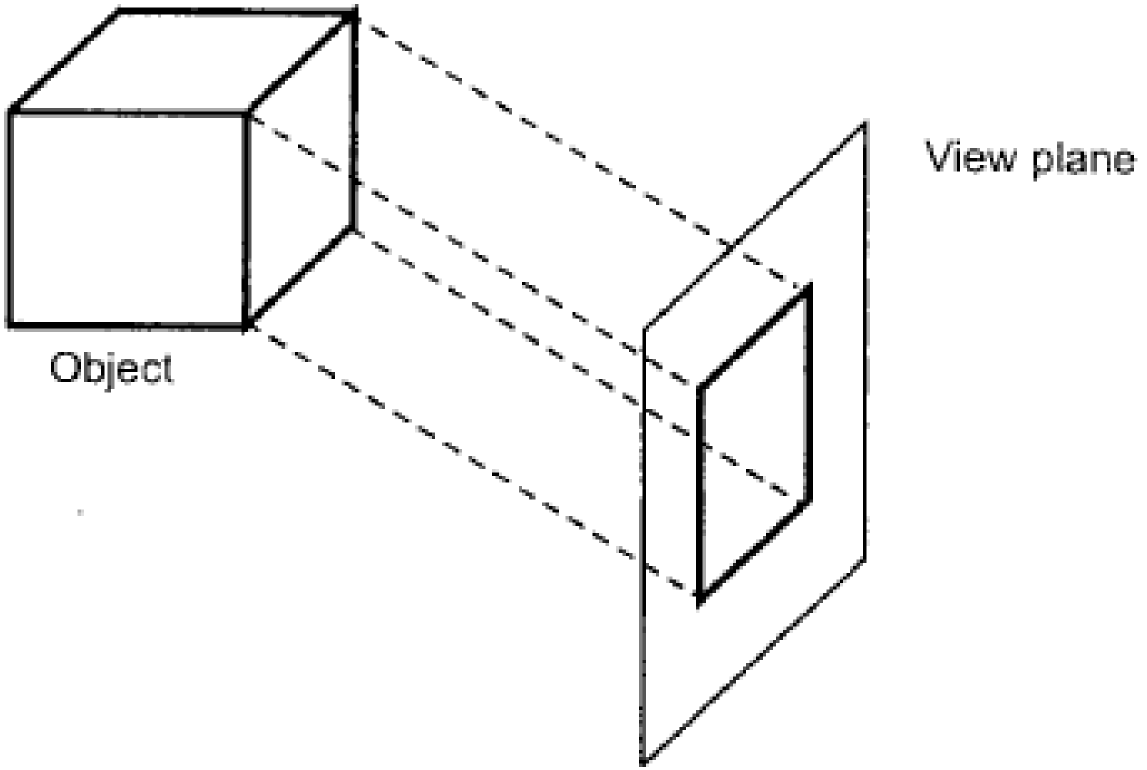


Fig. 6.5 Rotation about y axis

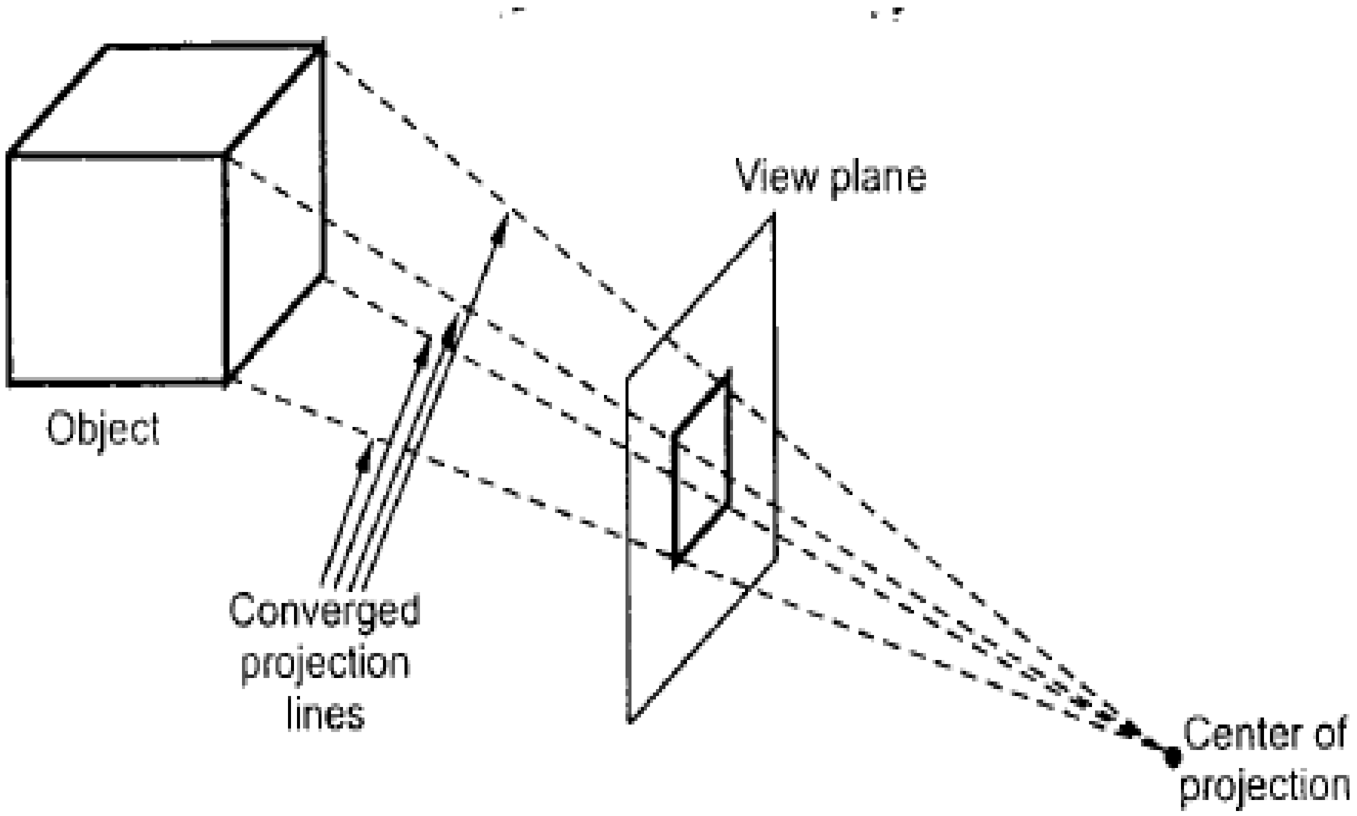
Projections

- Parallel Projection

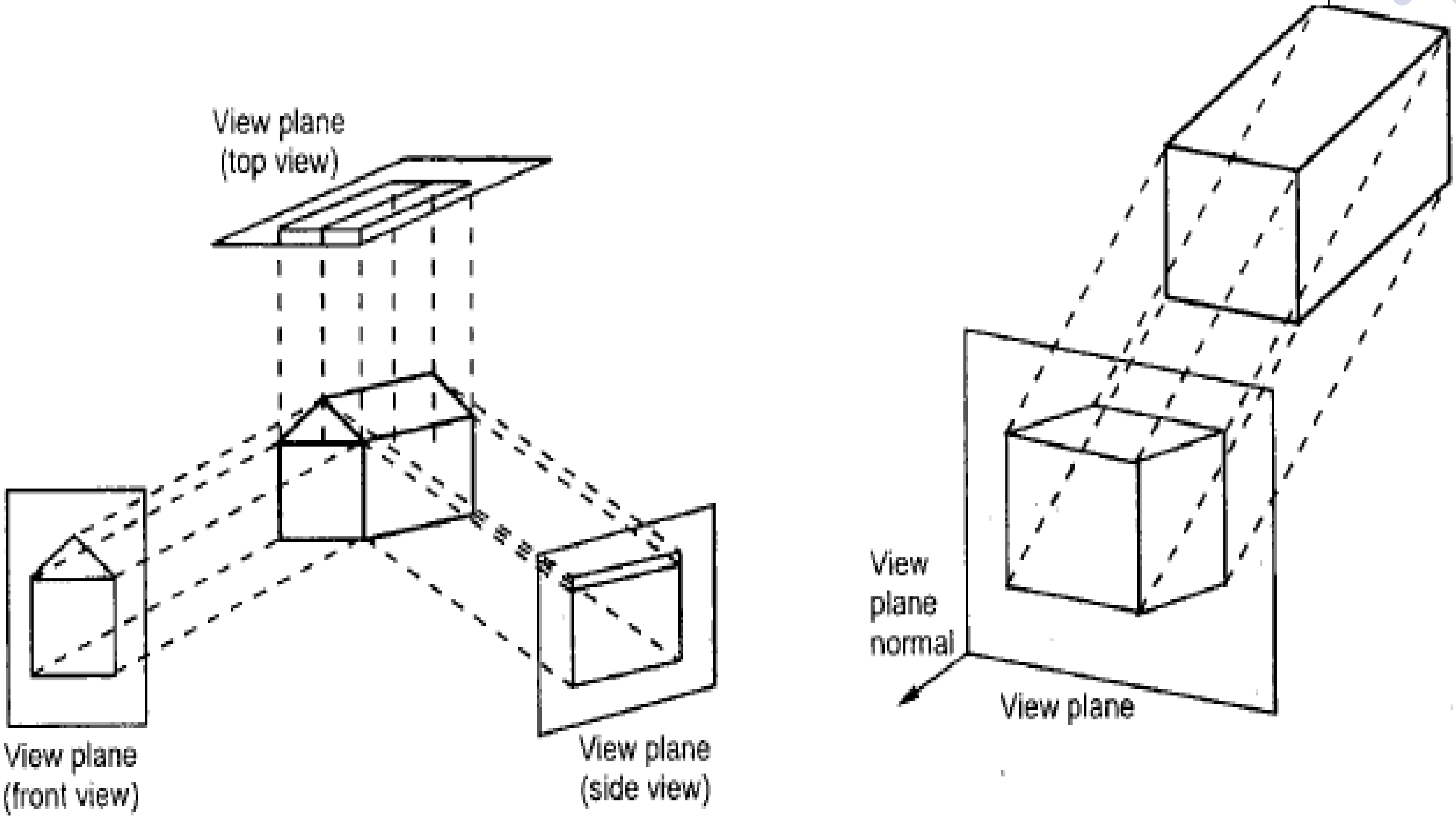


Projections (contd…)

- Perspective Projection



Types of Parallel Projections



(a) Orthographic parallel projection

(b) Oblique parallel projection