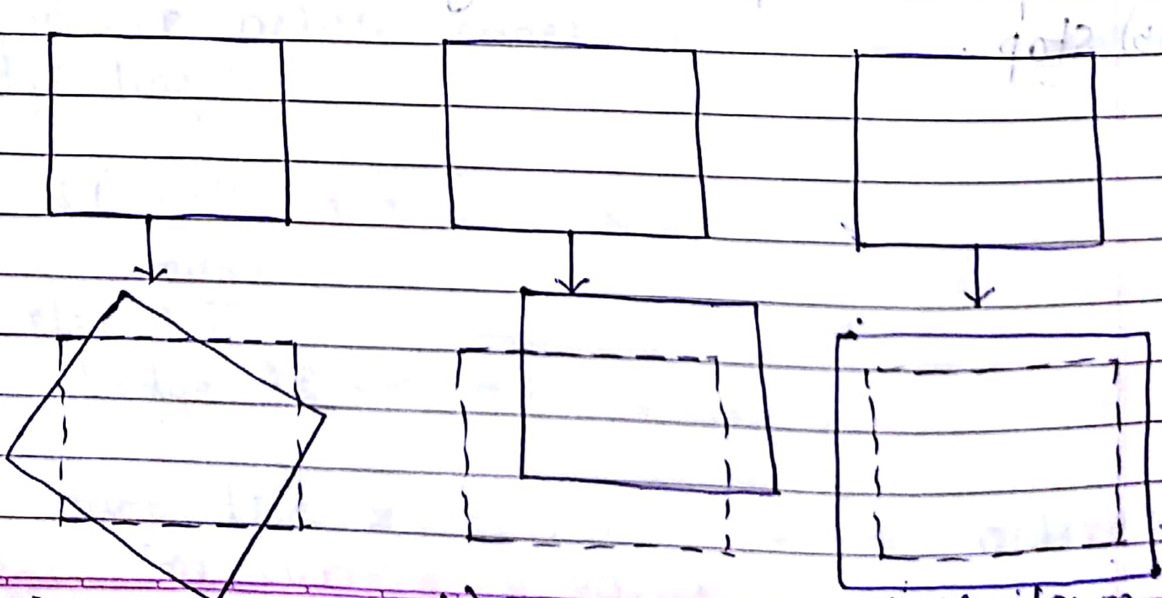
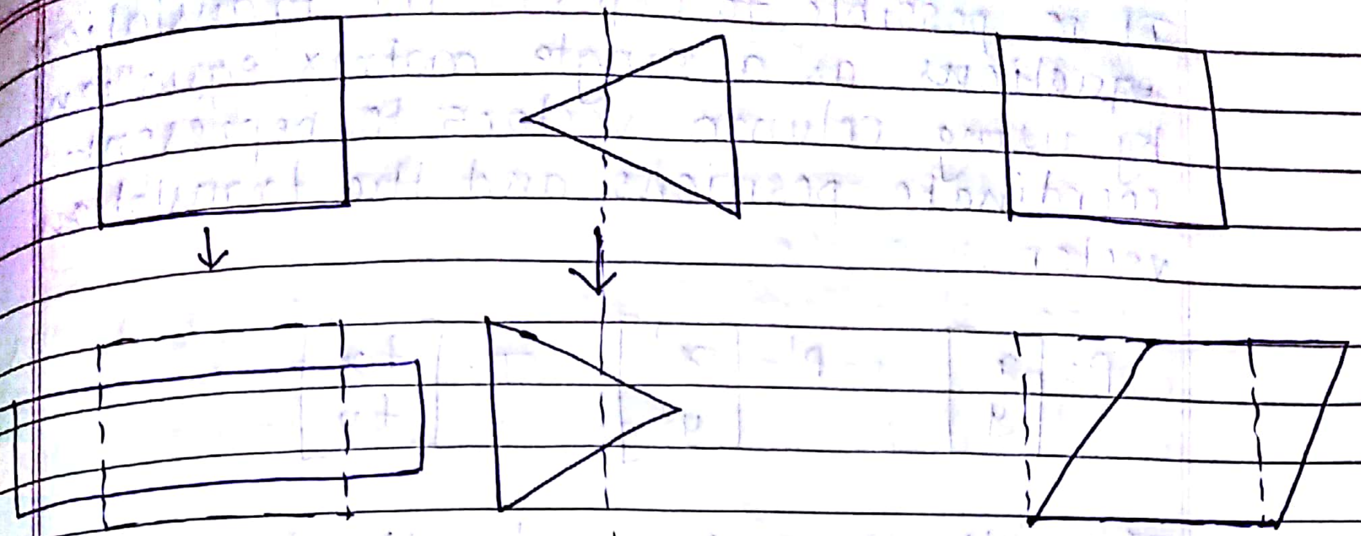
Chapter - 3

Overview of Transformations:

- For many graphics systems, procedures and other output primitives with attributes are already availabe which we can use to draw variety of pictures on screen. But in many applications there is need for alterning or manipulating pictures on display.
- In animation applications one need to move object along some specified path. Such changes in orientation, size and shape of an object alter co-ordinates of object and this is done with geometric transformation.
- Basic geometric transformations are:-
  1) Translation
  2) Scaling
  3) Rotation.
- Other transformations are :-
  1) Reflection
  2) Shear



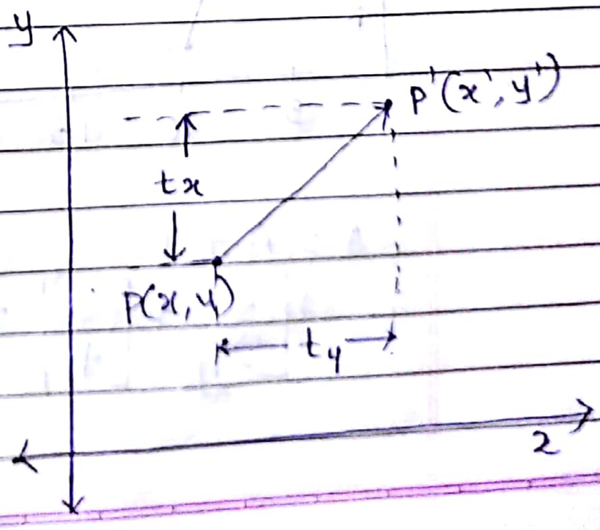
a) Rotation b) Translation c) Uniform Scaling.



b) Non-Uniform Scaling. e) Reflection f) Shearing.

① Translation :-

Translation is the process of changing the position of an object in a straight line path from one co-ordinate location to another. We can translate a two dimensional point by adding translation distances, t_x and t_y, to the original coordinate position (x, y) to move the point to a new position (x', y') as shown in fig.



$$x' = x + t_x$$
$$y' = y + t_y$$

The translation distance pair (t_x, t_y) is called as translation vector or shift vector.

It is possible to express the translation equations as a single matrix equation by using column vectors to represent coordinate positions and the translation vector :-

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

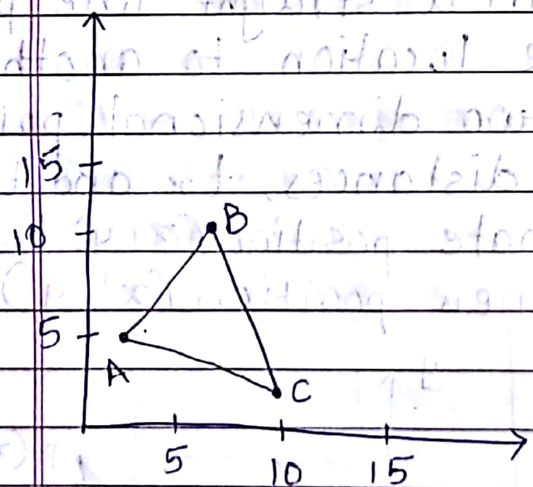
This allows us to write the two dimensional translation equations in the matrix forms :-

$$P' = P + T$$

Problem :-

Translate a polygon with coordinates A(2, 5), B(7, 10) and C(10, 2) by 3 units in x direction and 4 units in y direction.

⇒



$$A' = A + T$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$B' = B + T$

$$= \begin{bmatrix} 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

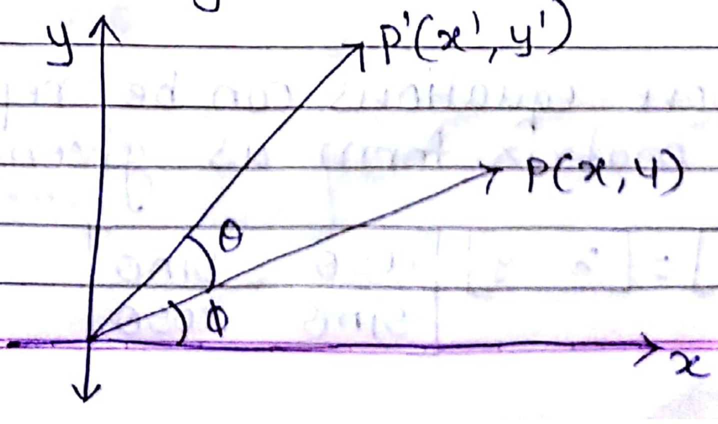
$C' = C + T$

$$= \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

Rotation :-

A two dimensional rotation is applied to an object by repositioning it along a circlular path in the xy plane.
To generate a rotation, we specify a rotation angle θ and the position of the rotation point about which the object is to be rotated.
Let us cosider the rotation of the object about the origin.



Here, r is the constant distance of the point from the origin, angle ϕ is the original angular position of the point from the horizontal, angle θ is the rotation angle.
Using standard trigonometric equations, we can express the transformed coordinates in terms of an angle θ and ϕ as,

$$\left.\begin{aligned} x' &= r\cos(\phi+\theta) \\ &= r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' &= r\sin(\phi+\theta) \\ &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{aligned}\right\} (1)$$

The original co-ordinates of the point in the polar co-ordinates are given as :-

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} (2)$$

Substituting eq.^{n} (2) in (1), we get transformation equations for rotating a point (x, y) through an angle θ about the origin as :-

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

The above equations can be represented in the matrix form as given below :-

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$P' = P \cdot R$$

where R is the rotation matrix and is given as,

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

This equation is for the counterclockwise or anticlockwise rotation.

For negative values of θ i.e. clockwise rotation, the rotation matrix becomes.

$$R = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \because \cos(-\theta) = \cos\theta, \ \sin(-\theta) = -\sin\theta$$

Problem:-

A point (4,3) is rotated counter clockwise by an angle 45°. Find the rotation matrix and the resultant point.

⇒

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P' = \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4\sqrt{2} - 3\sqrt{2} & 4\sqrt{2} + 3\sqrt{2} \end{bmatrix}$$

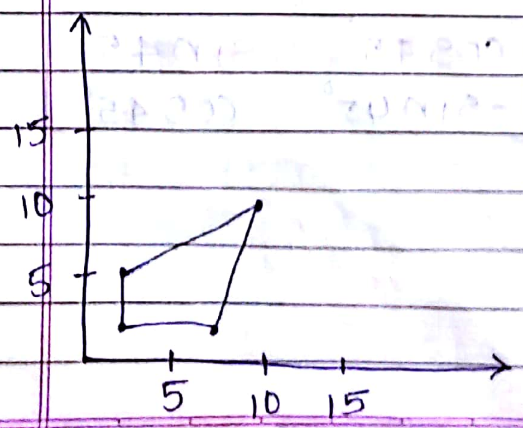
$$= \begin{bmatrix} 1/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}$$

Scaling :-

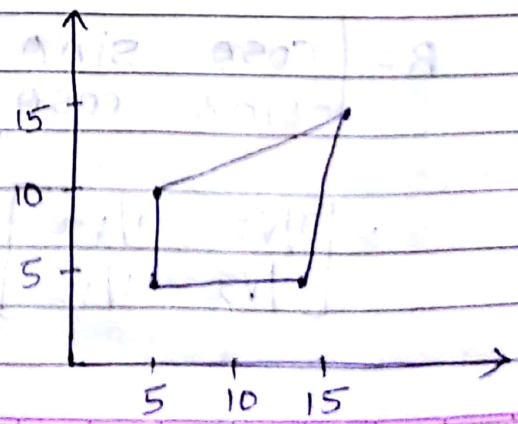
A scaling transformation changes the size of an object.
This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors S_x and S_y to produced the transformed coordinates (x', y').

$$x' = x \cdot S_x$$
$$y' = y \cdot S_y$$

Scaling factors S_x scales object in the x direction and scaling factor S_y scales object in the y direction.



(a)



(b)

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

$$= \begin{bmatrix} x \cdot S_x & y \cdot S_y \end{bmatrix}$$
$$= P \cdot S$$

Any positive numeric values are valid for scaling factors S_x and S_y.
Values less than 1 reduce the size of the objects and values greater than 1 produce an enlarged object.
For both S_x and S_y values equal to 1, the size of object does not change.
To get uniform scaling it is necessary to assign same value S_x and S_y.
Unequal values for S_x and S_y result in differential scaling.

Problem :-

Scale the polygon with coordinates A(2,5), B(7,10) and C(10,2) by two units in x direction and two units in y direction.

⇒ $S_x = 2$ and $S_y = 2$

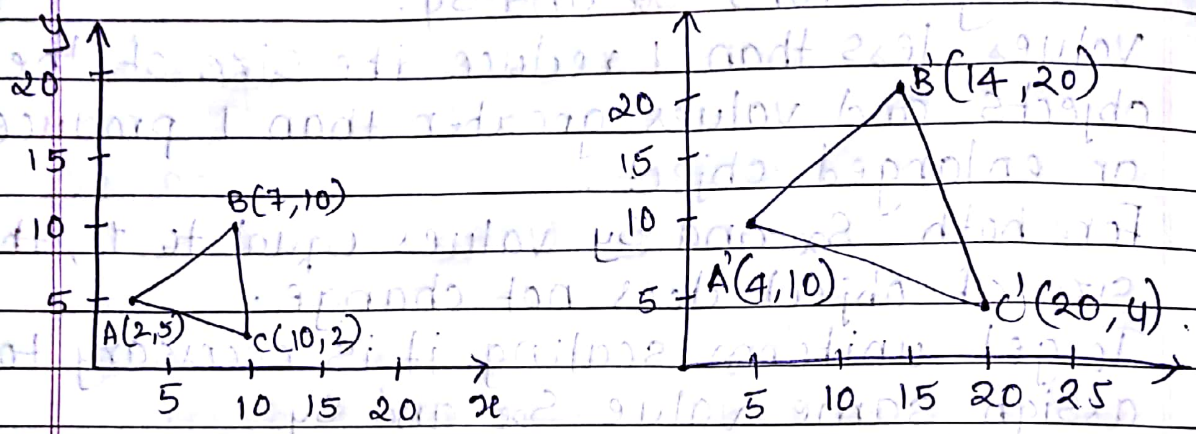
$$S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The object matrix is :-

$$\begin{array}{c|cc} & x & y \\ \hline A & 2 & 5 \\ B & 7 & 10 \\ C & 10 & 2 \end{array}$$

$$\begin{array}{c} A' \\ B' \\ C' \end{array}\begin{bmatrix} x_1' & y_1' \\ x_2' & y_2' \\ x_3' & y_3' \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 10 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 14 & 20 \\ 20 & 4 \end{bmatrix}$$



a) Original Object b) Scaled Object.

Homogeneous Co-ordinates :-

To produce a sequence of transformations with above equations, such as translation followed by rotation and then scaling, we must calculate the transformed coordinate one step at a time.
First co-ordinates are translated, then these translated coordinates are scaled and finally, the scale coordinates are rotated. But this sequential transformation process is not efficient.

A more efficient approach is to combine sequence of transformations into one transformation so that the final coordinate positions are obtained directly from initial co-ordinates.

In order to combine sequence of transformations we have to eliminate the matrix addition associated with the translation terms in M_2. To achieve this we have to represent matrix M_1 in 3x3 matrix instead by 2x2 introducing an additional dummy coordinate w. Here, points are specified by three numbers instead of two.

This coordinate system is called homogeneous coordinate system and it allows us to express all transformation equations as matrix multiplication.

The homogeneous coordinate is represented by a Triplet (X_w, Y_w, W).

For two dimensional transformations, we can have the homogeneous parameter W to be any non zero value. But it is convenient to have $w=1$.

Therefore, each two dimensional position can be represented with homogeneous coordinate as $(x, y, 1)$

Homogeneous coordinates for Translation:-

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

Therefore we have

$$[x' \; y' \; 1] = [x \; y \; 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$= [x + t_x \; \; y + t_y \; \; 1]$$

Homogeneous coordinates for rotation:-

The homogeneous coordinates for rotation are

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore we have,

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta & 1 \end{bmatrix}$$

Homogeneous coordinates for scaling :-

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore we have,

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

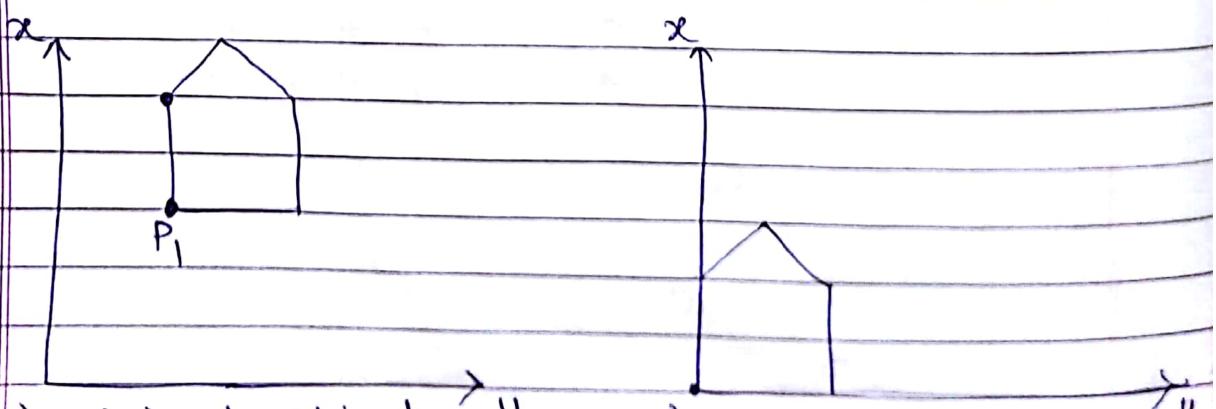
Composite 2D Transformations :-

The purpose of composing transformation is to gain efficiency by applying a single composed transformation to a point rather than applying a series of transformations, one after the other.

Rotation about an arbitrary point (Pivot Point):-

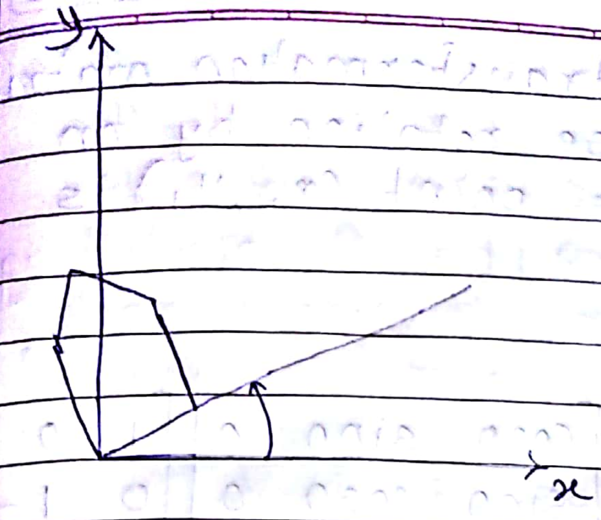
To do rotation of an object so that about any selected arbitrary point $P_1(x_1, y_1)$, following sequence of operations shall be performed:

1) Translate :- Translate an object so that arbitrary point P_1 is moved to coordinate origin.
2) Rotate :- Rotate object about origin.
3) Translate :- Translate object so that arbitrary point P_1 is moved back to its original position.

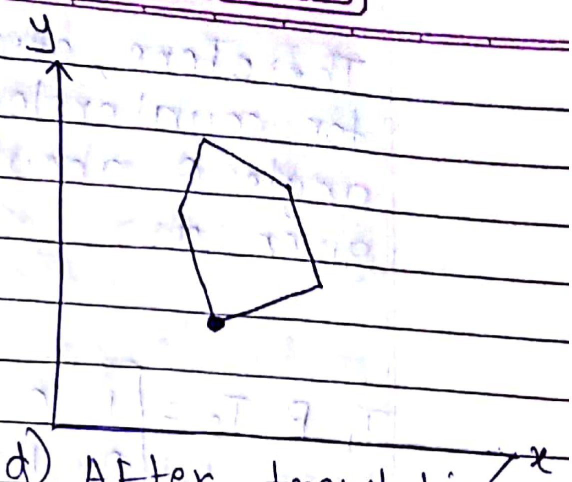


a) Original Object

b) After translating P_1 to origin



c) After Rotation



d) After translation to origin P_1

The translation matrix to move point (x_1, y_1) to the origin is

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -y_1 & 1 \end{bmatrix}$$

The rotation matrix for counterclockwise rotation of point about the origin is given as

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation matrix to move the point back to its original position is

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 & y_1 & 1 \end{bmatrix}$$

Therefore, overall transformation matrix for counterclockwise rotation by an angle θ about the point (x_1, y_1) is given as,

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -y_1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 & y_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_1\cos\theta + y_1\sin\theta & -x_1\sin\theta - y_1\cos\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 & y_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_1\cos\theta + y_1\sin\theta + x_1 & -x_1\sin\theta - y_1\cos\theta + y_1 & 1 \end{bmatrix}$$

Scaling about an Arbitrary Point :-

⇒ Scaling about an arbitrary point follows sequence of operations as follows :-

Scale object around point $P_1(x_1, y_1)$

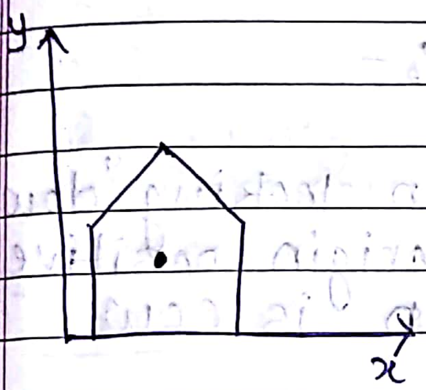
1) Translate P_1 to origin
2) Scale at origin
3) Translate back to P_1.

Equation for this in composite transformation matrix form is:

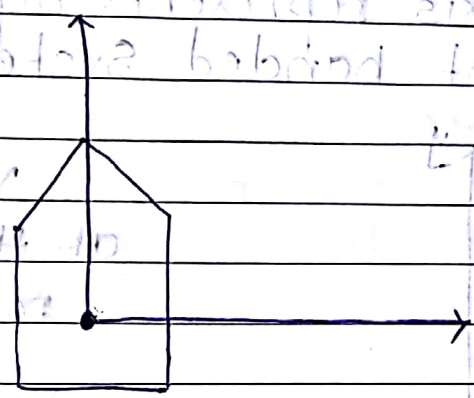
$$P' = T(x_1, y_1)\, S \cdot (S_x\, S_y) \cdot T(-x_1, -y_1) \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

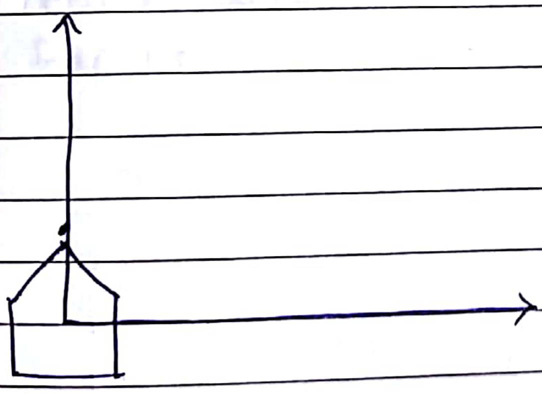
$$= \begin{bmatrix} S_x & 0 & x_1(1-S_x) \\ 0 & S_y & y_1(1-S_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



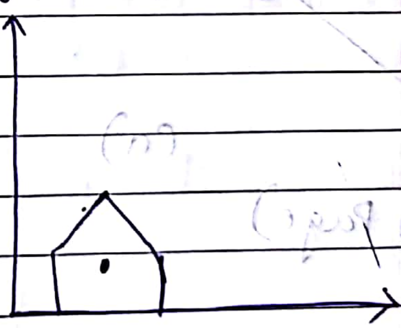
a) Original House



b) Translate P_1 to origin.



c) Scale



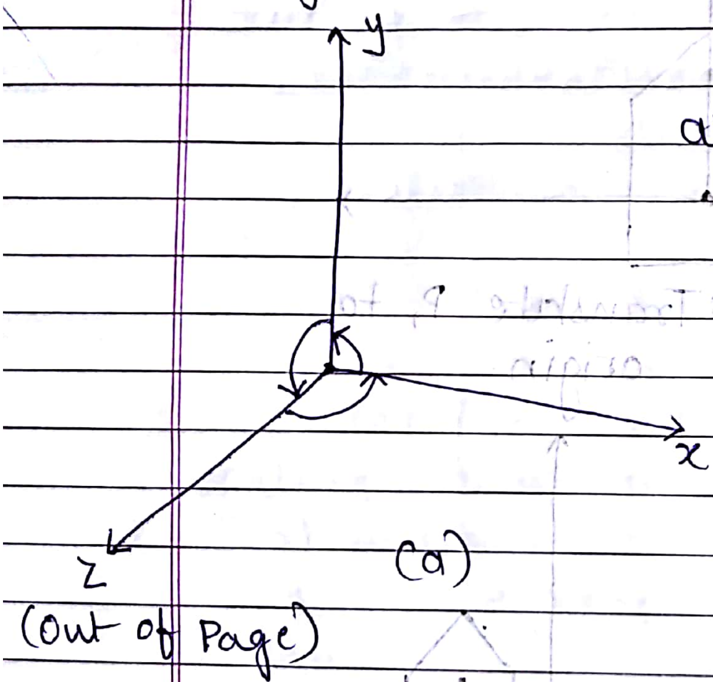
d) Translate to original position.

3D Transformations :-

Three dimensional (3D) geometric transformations and object modelling are extended from two dimensional methods by including z co-ordinates.

Representations of 3D Transformations :-

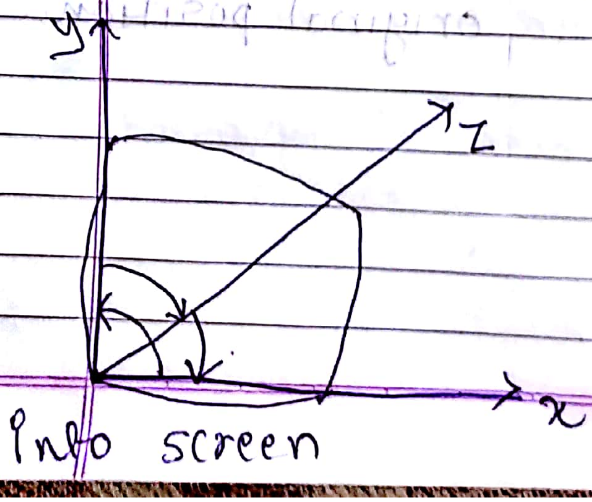
- z axis represents depth
- Right handed System :-



(a)

When looking "down" at the origin positive rotation is CCW.

- Left Handed System :-



When looking down positive rotation is in CW.

3D Homogeneous Co-ordinates :-

- Homogeneous co-ordinates for 2D space requires 3D vectors and matrices.
- Homogeneous co-ordinates for 3D space requires 4D vectors and matrices.

∴ A point is represented by $[x, y, z, w]$

- Like 2D, basic transformations in 3D also are :
  1) Translation
  2) Scaling
  3) Rotation.

Translation :-

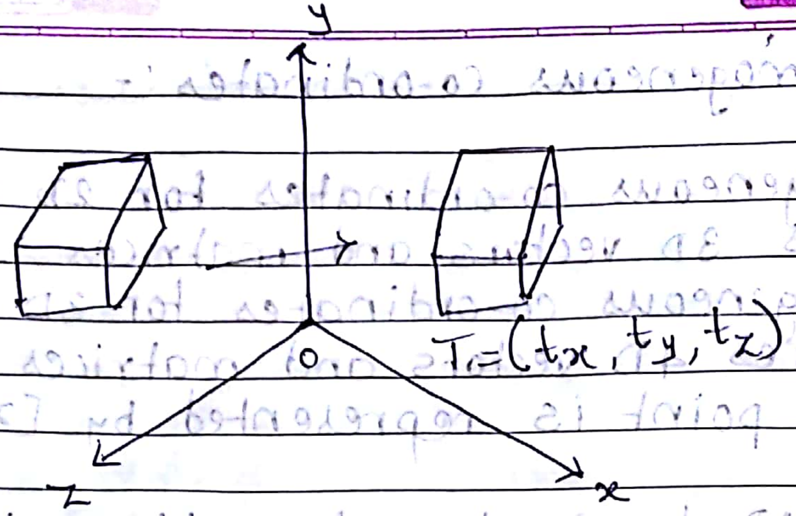
Three dimensional transformation matrix for translation with homogeneous coordinates with their own translation factor.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

$$P' = P \cdot T$$

$$\therefore [x' \;\; y' \;\; z' \;\; 1] = [x \;\; y \;\; z \;\; 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

$$= [x+t_x \;\; y+t_y \;\; z+t_z \;\; 1]$$

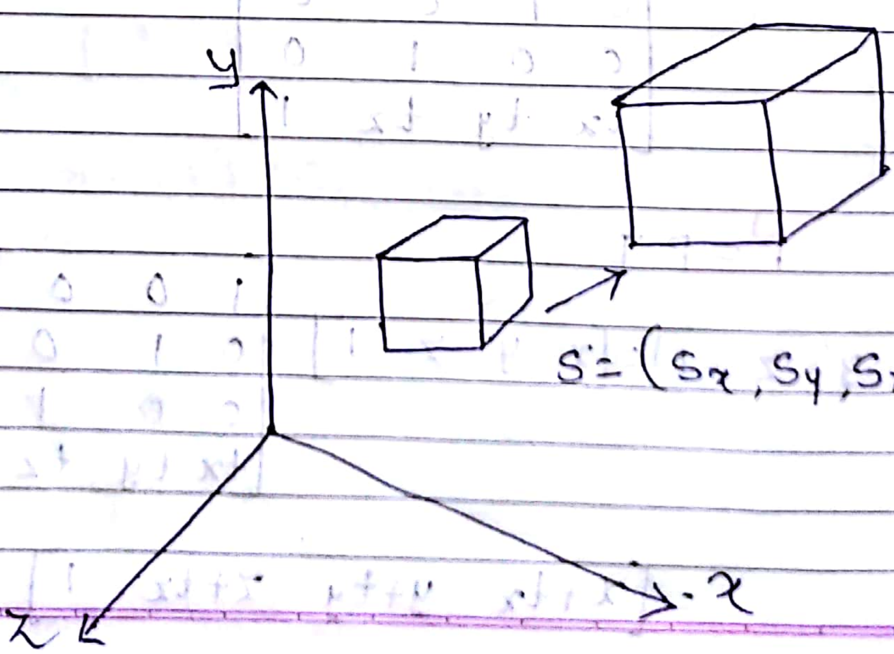


2) Scaling :-

Three dimensional transformation matrix for scaling with homogeneous coordinates is :

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$P' = P \cdot S$

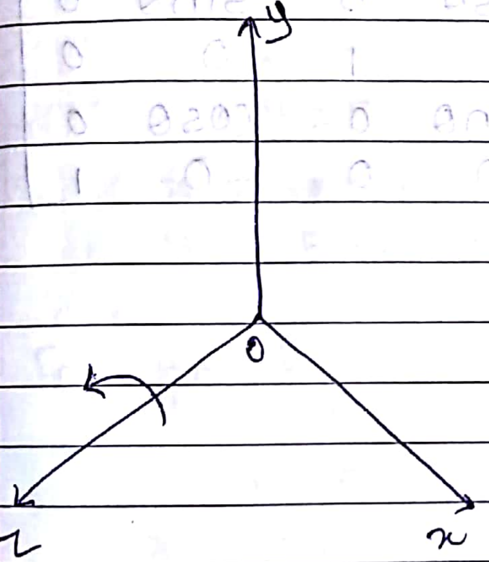


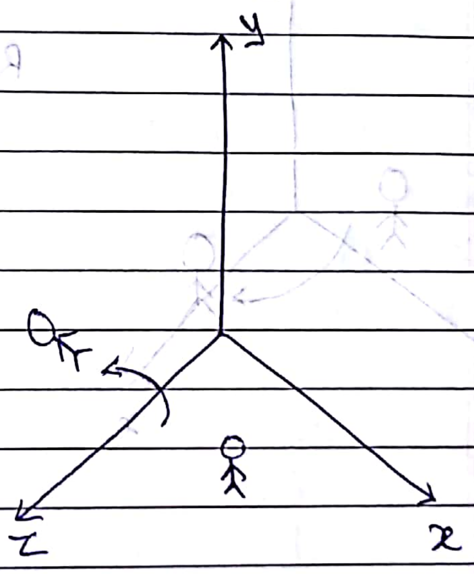
$$[x' \quad y' \quad z' \quad 1] = [x \quad y \quad z \quad 1] \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [x \cdot S_x \quad y \cdot S_y \quad z \cdot S_z \quad 1]$$

Rotation :-

Rotation about an z-axis :-

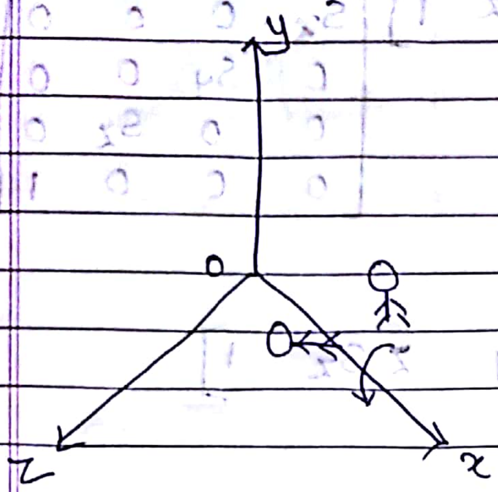




$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

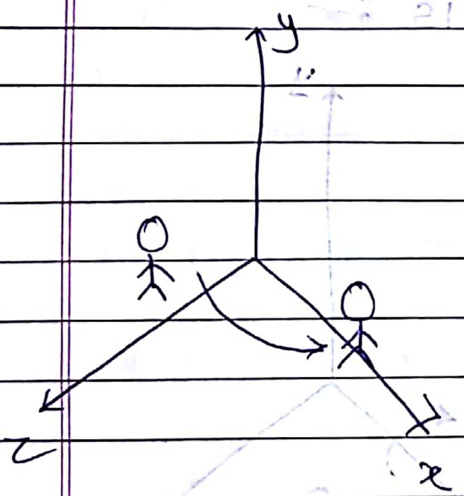
The positive value of angle θ indicates counterclockwise rotation.
For clockwise rotation value of angle θ is negative.

Rotation about an x-axis :-



$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about an y-axis :-



$$R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem :-

1) Translate a triangle ABC by 5 units in x direction where co-ordinates are A(5,5), B(10,5), C(10,10).

⇒

$x' = x + t_x$

$y' = y + t_y$

$t_x = 5 \qquad t_y = 0$

For point A,

$x = 5$

$y = 5$

$x' = \cancel{10}\ 5 + 5 = 10$

$y' = 5 + 0 = 5$

For point B

$x = 10$

$B = 5$

$x' = 10 + 5 = 15$

$y' = 5 + 0 = 5$

For point C

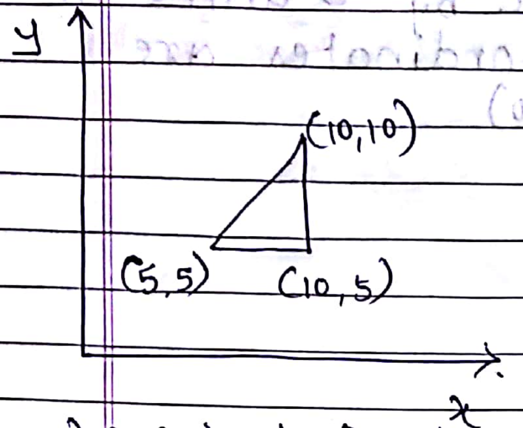
$x = 10$

$y = 10$

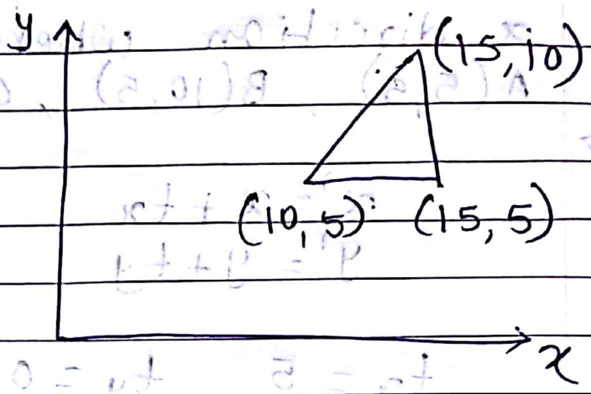
$x' = 10 + 5 = 15$

$y' = 10 + 0 = 10$

$A' = (x', y') = (10, 5)$
$B' = (x', y') = (15, 5)$
$C' = (x', y') = (15, 10)$



a) Original Position



b) After Translation

2) Scale the triangle ABC to reduce it to half of its size where coordinates of triangle are :
A(5,5) B(10,5) C(10,10).

⇒

Given :-

$A = (x, y) = (5, 5)$
$B = (x, y) = (10, 5)$
$C = (x, y) = (10, 10)$

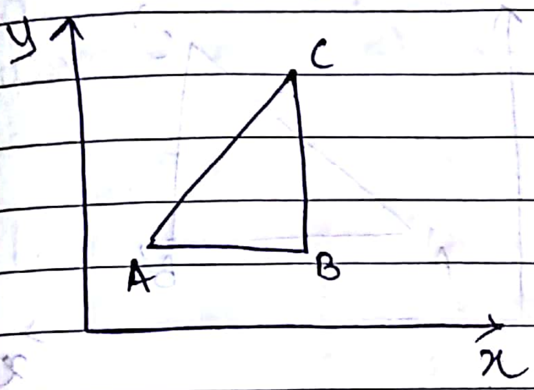
$S_x = 0.5$

$S_y = 0.5$

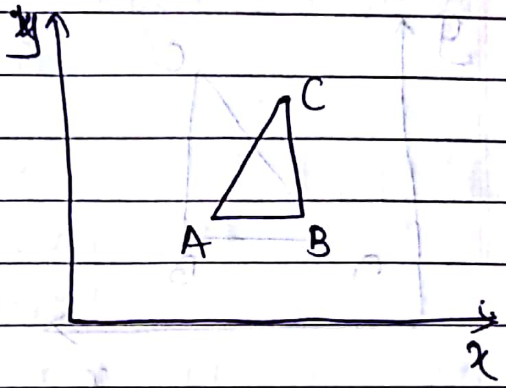
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 5 & 5 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & 5 & 5 \\ 2.5 & 2.5 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

$A' = (x', y') = (2.5, 2.5)$
$B' = (x', y') = (5, 2.5)$
$C' = (x', y') = (5, 5)$



a) Before Scaling



b) After Scaling.

3) Scale same above triangle ABC to extended it to double of its size only in x-direction.

→

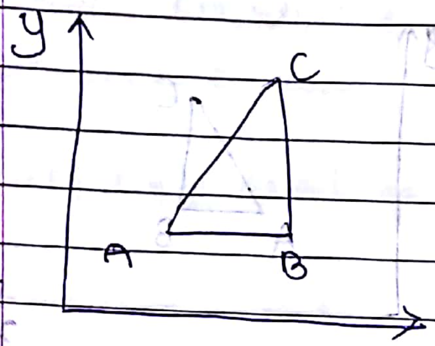
$A(x, y) = (5, 5)$
$B(x, y) = (10, 5)$
$C(x, y) = (10, 10)$

$S_x = 2$
$S_y = 1$

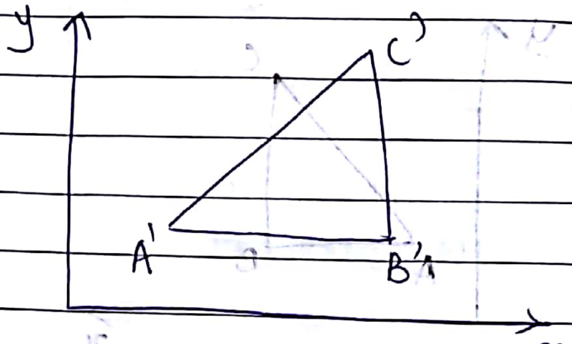
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 5 & 5 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 10 & 20 & 20 \\ 5 & 5 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$A' = (x', y') = (10, 5)$
$B' = (x', y') = (20, 5)$
$C' = (x', y') = (20, 5)$



a) After Scaling.



b) Before Scaling

4) A point (4,3) is rotated in counterclockwise by an angle 45°. Find the rotation matrix and the resultant point.

⇒

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P' = \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4\sqrt{2} - 3\sqrt{2} & 4\sqrt{2} + 3\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}$$

5) Rotate a line AB in counterclockwise direction for 45° angle about an origin where $A(5,5)$ and $B(20,5)$.

→

Given,

$A = (x, y) = (5, 5)$

$B = (x, y) = (20, 5)$

$\theta = 45^\circ$ in counterclockwise rotation

Solving by homogeneous co-ordinate matrix for rotation about origin.

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 5 & 1 \\ 20 & 5 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 5 & 1 \\ 20 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 5 & 1 \\ 20 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10/\sqrt{2} & 1 \\ 15/\sqrt{2} & 40/\sqrt{2} & 1 \end{bmatrix}$$

$$A' = (x', y') = \left(0, \frac{10}{\sqrt{2}}\right)$$

$$B' = (x', y') = \left(\frac{15}{\sqrt{2}}, \frac{40}{\sqrt{2}}\right)$$

6) Give 3x3 homogeneous coordinate transformation matrix for each of the following translations:-
a) Shift the image to the right by 3 units
b) Shift the image up by 2 units
c) Move the image down by 1/2 units and right by 1 unit
d) Move the image down 2/3 unit and left by 4 units.

⇒ Homogeneous coordinates for translation are :-

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

a) Here, $t_x = 3$, $t_y = 0$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

b) $t_x = 0$, $t_y = 2$ $\quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

c) $t_x = 1$, $t_y = -0.5$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}$$

d) $t_x = -4$ $t_y = -0.66$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -0.66 & 1 \end{bmatrix}$$

7) Find the transformation matrix that transforms the given square ABCD to half its size with centre still remaining at the same position. The coordinates of the square are: A(1,1), B(3,1), C(3,3), D(1,3) and centre at (2,2). Also find resultant coordinates of square.

⇒

This transformation can be carried out in following steps:-

1) Translate the square so that its center coincides with origin.
2) Scale the square with respect to the origin.
3) Translate square back to its original position.

$$T_1 \cdot S \cdot T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 1.5 & 1 \\ 2.5 & 1.5 & 1 \\ 2.5 & 2.5 & 1 \\ 1.5 & 2.5 & 1 \end{bmatrix}$$

$A' = (1.5, 2.5)$
$B' = (2.5, 2.5)$
$C' = (2.5, 2.5)$
$D' = (1.5, 2.5)$

8) Find a transformation of triangle: A(1,0), B(0,1), C(1,1) by
a) Rotating 45° about the origin and then translating one unit in x and y direction.
b) Translating one unit in x and y direction and then rotating 45° about the origin.

⇒ Rotation matrix is:

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation matrix is:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R \cdot T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2}+1 & 1/\sqrt{2}+1 & 1 \\ -1/\sqrt{2}+1 & 1/\sqrt{2}+1 & 1 \\ 1 & \sqrt{2}+1 & 1 \end{bmatrix}$$

b) $T.R. = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{2} & 1 \\ -1/\sqrt{2} & 3/\sqrt{2} & 1 \\ 0 & 2\sqrt{2} & 1 \end{bmatrix}$$

9) Perform counterclockwise 45° rotation of triangle A(2,3), B(5,5), C(4,3) about point (1,1).

⇒

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_1\cos\theta + y_1\sin\theta + x_1 & -x_1\sin\theta - y_1\cos\theta + y_1 & 1 \end{bmatrix}$$

Here, $\theta = 45^0$, $x_1 = 1$ and $y_1 = 1$

Substituting values we get,

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & -\sqrt{2}+1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 5 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & -\sqrt{2}+1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2}+1 & \frac{3}{\sqrt{2}}+1 & 1 \\ 1 & 8/\sqrt{2}+1 & 1 \\ \frac{1}{\sqrt{2}}+1 & \frac{5}{\sqrt{2}}+1 & 1 \end{bmatrix}$$

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10) Apply the shearing transformation to square with A(0,0), B(1,0), C(1,1) and D(0,1) as given below:

a) Shear parameter value of 0.5 relative to the line $y_{ref} = -1$

b) Shear ~~to~~ parameter value of 0.5 relative to the line $x_{ref} = -1$

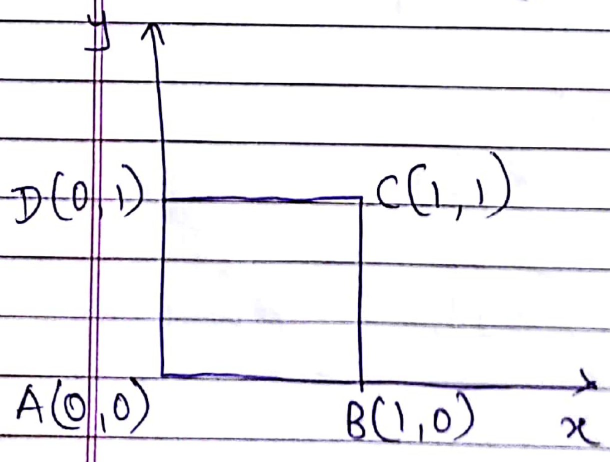
⇒

a) $Sh_x = 0.5 \quad y_{ref} = -1$

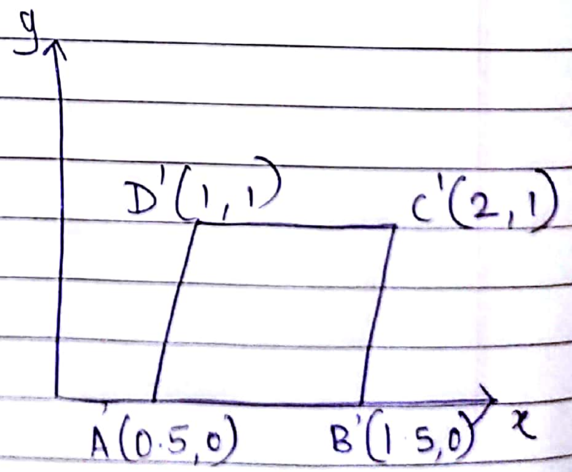
$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ -Sh_x \cdot y_{ref} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 1 \\ 1.5 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



a) Original square



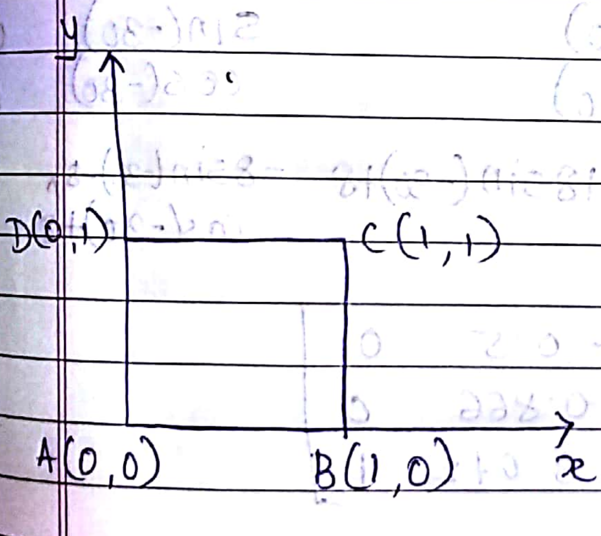
b) sheared square

b) $Sh_y = 0.5$ and $x_{ref} = -1$

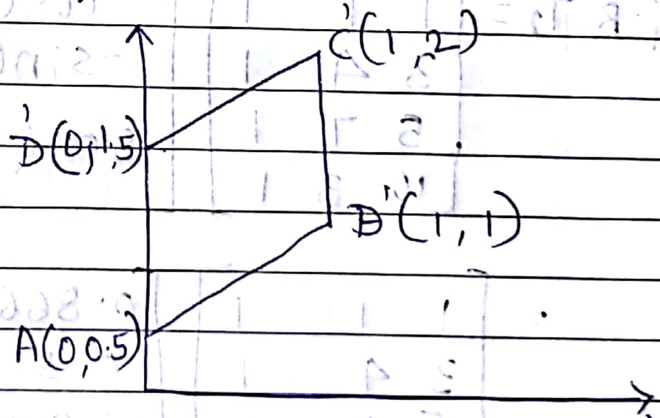
$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \cdot \begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & -Sh_y \cdot x_{ref} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.5 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1.5 & 1 \end{bmatrix}$$



a) Original Square



b) Sheared Square

11) Find out the final coordinates of a figure bounded by the coordinates (1,1) (3,4) (5,7) (10,3) when rotated about a point (8,8) by 30° clockwise direction and scaled by two units in x-direction and three units in y-direction.

⇒

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_1\cos\theta + y_1\sin\theta + x_1 & -x_1\sin\theta - y_1\cos\theta + y_1 & 1 \end{bmatrix}$$

In this case, it is clockwise rotation, therefore we take value of θ negative.

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 7 & 1 \\ 10 & 3 & 1 \end{bmatrix} \begin{bmatrix} \cos(-30) & \sin(-30) & 0 \\ -\sin(-30) & \cos(-30) & 0 \\ -8\cos(-30)+8\sin(-30)+8 & -8\sin(-30)-8\times\cos(-30)+8 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 7 & 1 \\ 10 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ -2.928 & 5.072 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1.562 & 5.438 & 1 \\ 1.67 & 7.036 & 1 \\ 4.902 & 8.634 & 1 \\ 7.232 & 2.67 & 1 \end{bmatrix}$$

$$T_1 \cdot R \cdot T_2 \cdot S = \begin{bmatrix} -1.562 & 5.438 & 1 \\ 1.67 & 7.036 & 1 \\ 4.902 & 8.634 & 1 \\ 7.232 & 2.67 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3.124 & 16.314 & 1 \\ 3.34 & 21.108 & 1 \\ 9.804 & 25.902 & 1 \\ 14.464 & 8.01 & 1 \end{bmatrix}$$

Programs :-

1) Program for 2D Translation

⇒

```
# include <stdio.h>
# include < conio.h>
# include < graphics.h>

void main()
{
  int gd=DETECT, gm;
  int x1,y1, x2, y2, tx, ty;

 initgraph(&gd, &gm, "c:\\turboc3\\bgi");

printf("Enter the original line coordinates");
scanf("%d%d%d%d", &x1, &y1, &x2, &y2);
line(x1, y1, x2, y2);
getch();
```

```
printf("Enter the translation factors:");
scanf("%d %d", &tx, &ty);

printf("Line after translation");

    x1 = x1 + tx;
    y1 = y1 + ty;
    x2 = x2 + tx;
    y2 = y2 + ty;

 line(x1, y1, x2, y2);
 getch();
 closegraph();
}
```

2) Program for 2D scaling.

⇒

```
 # include <graphics.h>
 # include <stdio.h>
 # include <conio.h>

 void main()
 {
int x1, y1, x2, y2;
float sx, sy;
int gd, gm;
detectgraph(&gd, &gm);
initgraph(&gd, &gm, "c:\\tc\\bgi");

printf("Enter the coordinates of original
        line");
scanf("%d %d %d %d", &x1, &y1, &x2, &y2)
```

```
line(x1, y1, x2, y2);
getch();

printf("Enter Scaling factors:");
scanf("%f%f", &Sx, &Sy);

    x1 = x1 * Sx;
    y1 = y1 * Sy;
    x2 = x2 * Sx;
    y2 = y2 * Sy;

printf("Line after scaling");
    line(x1, y1, x2, y2);
    getch();
    closegraph();
}
```

3) Program for 2D Rotation

→

```
#include<graphics.h>
#include< conio.h>
#include< stdio.h>

void main()
{
  int x1, x2, y1, y2, xn, yn
  double r11, r12, r21, r22, r, th;
  clrscr();
printf(" Enter the coordinates of line:");
scanf("%d%d%d%d", &x1, &y1, &x2, &y2);
  int gd, gm;
  detectgraph(&gd, &gm);
  initgraph(&gd, &gm, "c:\\tc\\bgi");
```

```
printf("Line before rotation:");
  line(x1,y1,x2,y2);
  getch();
printf("Enter the angle of rotation:");
 scanf("%f",&th);

r11=cos((th*3.14)/180);
r12= sin((th*3.14)/180);
r21=(-sin((th*3.14)/180));
r22=cos((th*3.14)/180);

  xn=((x2*r11)-(y2*r12));
  yn=((x2*r12)+(y2*r11));

 printf("Line after rotation");
  line(x1,y1,xn,yn);
 getch();
  closegrap();
}
}
```

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4) Program for 2D shearing

→

```
# include < graphics.h>
# include < conio.h>
# include < stdio.h>

void main()
{
  int x1, y1, x2, y2, x, y;
  int gd, gm;
  detectgraph(&gd, &gm);
  initgraph(&gd, &gm, "c:\\tc\\bgi");

  printf("Enter the coordinates of
    rectangle");
  scanf("%d %d %d %d", &x1, &y1, &x2, &y2);

      rectangle(x1, y1, x2, y2);
      getch();

 printf("Enter the x shear coordinates:");
  scanf("%d", &x);
     rectangle(x1, y1, x2*x, y2);
     getch();
 printf("Enter the y shear coordinates:");
   scanf("%d", &y);
      rectangle(x1, y1, x2, y2*y);
      getch();
      closegraph();
   }
```

Difference between Parallel and Perspective Projections:-

Parallel Projection	Perspective Projection
1) In parallel projection the centre of projection is at infinity.	1) In perspective projection, the centre of projection is at a finite distance.
2) Here, all projections are parallel to each other.	2) Here, projectors are not parallel.
3) It is a less realistic view.	3) It is more realistic
4) It is used for the applications where exact measurement is required.	4) It resembles to that of our photographic systems and human eye.
5) Eg:- use of drawing schematic diagrams.	5) eg:- Use in architectural rendering realistic views.